Brilliant: Vector Calculus

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Note: Latex reference: http://tug.ctan.org/info/undergradmath/undergradmath.pdf

1 Chapter 2: Basics, Vector fields

1.1 Chapter 2.1: Calculus of Motion

Consider vectors of motion against t of the form $\vec{x}(t) = \langle x(t), y(t), \ldots \rangle$.

- A **line** through $p = (a, b, c)$ parallel to $\vec{v} = \langle v_x, v_y, v_z \rangle$ is $\vec{x}(t) = \vec{p} + t\vec{v}$
- **velocity** is characterized completely by $\vec{v}(t) = \vec{x}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- The **speed** of an object along that line versus t is the length of v ($\|v\|$)
- Therefore, the speed of an object along line

  $$\langle x(t), y(t), z(t) \rangle = \langle 0, 2, -3 \rangle + t\langle 1, -2, 2 \rangle$$

  is

  $$\sqrt{1^2 + (-2)^2 + 2^2} = 3$$

- Note that $\vec{v}$ need not be constant. The speed of

  $$\vec{x}(t) = \vec{p} + 3\sin(2\pi t)\hat{u}, \|\hat{u}\| = 1$$

  would then be

  $$\|6\pi\cos(2\pi t)\hat{u}\| = |6\pi\cos(2\pi t)|$$

- **Acceleration** $a(t) = v'(t) = x''(t)$ is straightforward. Acceleration of

  $$x(t) = \langle -1 + \cos(t), 1, \cos(t) \rangle = \langle -\cos(t), 0, -\cos(t) \rangle$$

- An example position vector for a planet of distance r from the sun could be $\langle r\cos(t), r\sin(t)\rangle$. The acceleration vector points in the opposite direction: $\langle -r\cos(t), -r\sin(t)\rangle$. In addition to being the analytical second derivative, consider that the *force* of gravity, (which, by $F = ma$ is proportional to acceleration) points towards the sun, *with acceleration perpendicular to velocity.*
- A **helix** could be a 3D extension like $\langle r\cos(t), r\sin(t), b\cdot t\rangle$.

2 Chapter 2.2: Space Curves

- Note that while $\vec{x}(t) = \langle\cos(t), \sin(t), 5\rangle$ and $\vec{x}(t) = \langle\cos(2t), \sin(2t), 5\rangle$ describe the same curve, the space curve also records motion in time, so the *velocity* may be different.
- If $\vec{x}(t) = t\vec{v}$, then speed is $\frac{\|\vec{x}(t+\Delta t)-\vec{t}\|}{\Delta t} = \|\vec{v}\|$, direction is $\frac{\vec{v}}{\|\vec{v}\|}$, and velocity $\vec{v}$ is the product of speed and direction.
- So $\vec{v}(t) = \lim_{\Delta t\to 0}\frac{\vec{x}(t+\Delta t)-\vec{x}(t)}{\Delta t} = \vec{x}'(t) = \frac{d\vec{x}}{dt} = \langle x'(t), y'(t), z'(t)\rangle$
- Neat conceptual result: any $y = f(x)$ can be made into $x(t) = \langle t, f(t)\rangle$, and then $v(t) = \langle 1, f'(t)\rangle$, which points along the tangent line at $\langle t, f(t)\rangle$.
- Note that dot product derivatives work like regular product: $[\vec{a}(t)\cdot\vec{b}(t)]' = \vec{a}'(t)\cdot\vec{b}(t) + \vec{a}(t)\cdot\vec{b}'(t)$, but the cross product does not work the same since $\frac{d}{dt}[a\times b] = a'\times b + a\times b'$, but since $a\times b' = -b'\times a$, can't switch the order to $a'\times b + b'\times a$ due to this non-commutativity.
- If

  $$\vec{x}(t) = \vec{p} + t\vec{v},$$

  calculating velocity with respect to origin becomes

  $$\frac{d}{dt}\|\vec{x}(t)\| = \frac{\vec{x}(t)\cdot\vec{x}'(t)}{\|\vec{x}(t)\|} = \frac{\vec{x}}{\|\vec{x}\|}\cdot\vec{v},$$

  after rewriting the distance formula and chugging through the chain rule.
- However, it becomes more clear when considering that $(\vec{v}\cdot\hat{x})\hat{x}$ is the projection of the velocity vector onto the position vector. So, the length of this is the rate of change of distance from origin!

3 Chapter 2.3: Integrals and Arc Length

- Integral of a vector function can be defined componentwise in a straightforward way: $\int_a^b \vec{x}(t) = \langle\int_a^b x(t), \int_a^b y(t), \int_a^b z(t)\rangle$

- Example: if ball launched from origin with velocity $\langle 1,2,3\rangle$ and acceleration $\langle 0,0,-1\rangle$, it lands at

$$\frac{dv}{dt}dt = \langle 0,0,-1\rangle \tag{1}$$
$$\int \frac{dv}{dt}dt = v = \langle C, D, -t+F\rangle = \langle 1,2,3\rangle = \langle 1,2,-t+3\rangle, t=0 \tag{2}$$
$$x = \int v = \langle t+K, 2t+M, -\frac{1}{2}t^2+3t+N\rangle, x(\vec{0}) = \langle 0,0,0\rangle \tag{3}$$
$$\vec{x}(t) = \langle t, 2t, 3t-\frac{1}{2}t^2\rangle \tag{4}$$
$$z(t) = 0 \to t = 6 \to \vec{x}(6) = \langle 6, 12, 0\rangle \tag{5}$$
$$\tag{6}$$

- Also, generalizing $ds = \sqrt{(dx)^2+(dy)^2}$, the length of an arc from point a to b approaches $\boxed{\int_a^b ds = \int_{t_a}^{t_b} \|x'(t)\| dt}$
- Example: a helix $\langle a\cos(\omega t), a\sin(\omega t), b\omega t\rangle$, parametrized by time t can be rewritten in terms of s, the arc length:

$$s = \int \|x'(t)\| dt \tag{7}$$
$$s = \int \sqrt{(-\omega a\sin(\omega t))^2 + (\omega a\cos(\omega t))^2 + (b\omega)^2}dt \tag{8}$$
$$s = |\omega| \int \sqrt{(a^2+b^2)}dt \tag{9}$$
$$s = |\omega| t\sqrt{a^2+b^2} \tag{10}$$

- *Note: It's weird to think of time in terms of length.* Could be analytically useful?

4 Chapter 2.4: Frenet Formulae

Main idea: Establish three new vectors $\hat{T}(s), \hat{N}(s), \hat{B}(s)$ that change as we move along a space curve, instead of $\vec{x}(t)$ that changes over an external "time" idea.

Remember that $s = \int_0^t \|\vec{x}'(\tilde{t})\| d\tilde{t}$, so $\frac{ds}{dt} = \|\vec{x}'(t)\|$.

4.1 $\hat{T}$: Vector tangent to space curve

- Remember arc length is $s = \int_0^t \|\vec{x}'(\tilde{t}) d\tilde{t}\|$

- $\hat{T}$ is just normalized grad: $\frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}$
- This implies $\boxed{\frac{d\vec{x}}{ds} = \hat{T}}$ since

$$
\begin{align}
s &= \int_0^t \|\vec{x}'(\tilde{t})d\tilde{t}\| \tag{11}\\
\frac{ds}{dt} &= \|\vec{x}(t)\| \tag{12}\\
\hat{T} = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} &= \frac{d\vec{x}}{dt} \cdot \frac{dt}{ds} \tag{13}\\
\hat{T} &= \frac{d\vec{x}}{ds} \tag{14}\\
\tag{15}
\end{align}
$$

- So this is how the space curve $\vec{x}$ changes as it moves along the curve at length s.
- It's normalized, so it's the same whether parameterized by t, s, or whatever.

4.2 $\hat{N}$: Normal to curve (perpendicular to $\hat{T}$)

Normal vectors are:

- $\vec{x}''(t)$ normalized as $\boxed{\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|} = \hat{N}}$
- The normal vector to the curve
- $\perp$ to $\hat{T}$ in direction of acceleration. So a multiple of acceleration vector.
- $\frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}$. The following sequence shows any unit length vector is perpendicular to its derivative.

$$
\begin{align}
\|\hat{T}\| &= 1 \tag{16}\\
d(\|\hat{T}\|^2) &= 0 \tag{17}\\
d(\|\hat{T}\|^2) = d(\hat{T} \cdot \hat{T}) &= \hat{T}(t) \cdot 2\hat{T}'(t) \tag{18}\\
\hat{T}(t) \cdot \hat{T}'(t) &= 0 \tag{19}
\end{align}
$$

- $\frac{\frac{d\hat{T}}{ds}}{\|\frac{d\hat{T}}{ds}\|}$ since it's the same as the above, but parametrized over s instead of t. Doesn't change the direction of the vector!

Example:if $\vec{x}(t) = \langle R\cos(\omega t), R\sin(\omega t), 0\rangle$, then:

- $\vec{a} = \frac{d^2\vec{x}}{dt^2}$ just by definition
- $\vec{a} = -\omega^2\vec{x}$ just by calculation
- $\hat{T}(t) = \langle -\sin(\omega t), \cos(\omega t), 0\rangle$
- $\|\hat{T}(t)\| = 1$
- $\hat{N} = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = \langle -\cos(\omega t), -\sin(\omega t), 0\rangle$
- So $\vec{a} = R\omega^2\hat{N}$ by these formulae.

This leads us to believe acceleration and $\hat{N}$, the normed derivative of $\hat{T}$ are related.

The part of acceleration $\vec{a}$ parallel to $\hat{T}$ is the projection $(\vec{a}\cdot\hat{T})\hat{T}$

The perpendicular part is then $\vec{a}$ minus that: $\vec{a} - (\vec{a}\cdot\hat{T})\hat{T}$

This also equals $(\frac{ds}{dt})^2\|\frac{d\hat{T}}{ds}\|\hat{N}$ because

$$\vec{x}' = \frac{dx}{dt} = T = \hat{T}\cdot\|\frac{dx}{dt}\| \tag{20}$$

$$s = \int_0^t \|\vec{x}'(t)\| \to \frac{ds}{dt} = \|\vec{x}'(t)\| \tag{21}$$

$\hat{N} = \frac{d\hat{T}}{ds}$ normalized, so

$$\vec{a} = \frac{d^2\vec{x}}{dt^2} = \frac{d}{dt}(\|\frac{dx}{dt}\|\frac{\frac{dx}{dt}}{\|\frac{dx}{dt}\|}) = \frac{d}{dt}(\|\vec{x}'(t)\|\hat{T}(t)) = \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + \|\vec{x}'(t)\|\frac{d\hat{T}}{dt} \tag{22}$$

$$= \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{ds}\frac{ds}{dt} \tag{23}$$

$$= \frac{d\|\vec{x}'(t)\|}{dt}\hat{T} + (\frac{ds}{dt})^2\|\frac{d\hat{T}}{ds}\|\hat{N} \tag{24}$$

This is "a = $\hat{T}$'s parallel part plus $\hat{T}$'s perpendicular (N) part", so the second term is $a_\perp$

4.3 $\hat{T}$ and $\hat{N}$

- Form a plane, since first, any normal vector's derivative is perpendicular to the vector
- κ is curvature: how much we're curving in that $T \times N$ plane.
- $\boxed{\kappa = \|\frac{d\hat{T}}{ds}\|}$

- Therefore, by the definition of $\hat{N} = \frac{d\hat{T}/ds}{\|d\hat{T}/ds\|}$, $\boxed{\frac{d\hat{T}}{ds} = \kappa \hat{N}}$ **(Frenet equation 1)**

Note that curvature $\kappa(s) = \|\frac{d\hat{T}}{ds}\|$ is geometric (depends on s, not time) and changes as $\hat{T}$ changes.

Example: Curvature of $\vec{x}(t) = \langle \cos(t), \sin(t), bt \rangle$

$$x'(t) = \langle -\sin(t), \cos(t), b \rangle \tag{25}$$

$$\|x'(t)\| = \sqrt{1+b^2} \tag{26}$$

$$s = \int_0^t \|x'(\tilde{t})\| d\tilde{t} = \int_0^t \sqrt{(1+b^2)} = t\sqrt{(1+b^2)} \rightarrow t = \frac{s}{\sqrt{1+b^2}} \tag{27}$$

Do the substitution of $\frac{s}{\sqrt{1+b^2}}$ for t above to get $x'(s)$, and from there, you can figure out $\frac{d\hat{T}}{ds}$ and normalize to get $\kappa = \frac{1}{1+b^2}$

4.4 $\hat{B}$ is binormal: perpendicular to both

- defined as $\boxed{\hat{B} = \hat{T} \times \hat{N}}$
- Therefore, by derivative

$$\frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \tag{28}$$

$$\frac{d\hat{B}}{ds} = \kappa \hat{N} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} \tag{29}$$

$$\frac{d\hat{B}}{ds} = \hat{T} \times \frac{d\hat{N}}{ds} \tag{30}$$

  This means T is orthogonal to dB, and we already know B and dB are orthogonal. We're working in 3D with the cross product, so dB is parallel to N.

- Therefore, we define **torsion** τ so that $\boxed{-\frac{d\hat{B}}{ds} = \tau \hat{N}}$ **(Frenet equation 2)**. Negative sign by convention.
- Can also cross by N on both sides to get $-\frac{d\hat{B}}{ds} \times \hat{N} = \tau$
- τ measures how the plane defined by $\hat{T}, \hat{N}$ twists around. On a circle, $\hat{B}$ wouldn't change, so the derivative would be zero.

- **Final Frenet equation**. Prereq: $\hat{B} = \hat{T} \times \hat{N} \to \hat{N} = \hat{B} \times \hat{T} \to \hat{T} = \hat{N} \times \hat{B}$

$$\frac{d\hat{N}}{ds} = \frac{d\hat{B}}{ds} \times \hat{T} + \hat{B} \times \frac{d\hat{T}}{ds} \tag{31}$$

$$\frac{d\hat{N}}{ds} = -\tau\hat{N} \times \hat{T} + \hat{B} \times \kappa\hat{N} \tag{32}$$

$$\frac{d\hat{N}}{ds} = \tau\hat{B} - \kappa\hat{T} \tag{33}$$

5 Chapter 2.5: Parametrized Surfaces

Main approaches to describing a surface:

- Can parameterize by $\vec{x}(u,v) = x(u,v), y(u,v), z(u,v)$
- Can perhaps parameterize $f(x,y,z) = c$ by $z = g(x,y)$
- Can also use ideas like $\nabla f = 0$ to find a normal.

There are many out-of-the-box paremetrizations including:

- Sphere at (0,0,0): $\vec{x}(u,v) = \langle R\cos(u)\sin(v), Rsin(u)\sin(v), R\cos(v)\rangle$, where $u \in [0, 2\pi), v \in [0, \pi]$
- Rotate function $y = f(x)$ around the x-axis: $\vec{x}(u,v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$, where $u \in D, v \in [0, 2\pi]$

Tangent vectors to $\vec{x}(u,v)$ are $\frac{\delta\vec{x}}{du}, \frac{\delta\vec{x}}{dv}$, so unit normal $\boxed{\hat{n} = \pm\frac{\frac{d\vec{x}}{du} \times \frac{\delta\vec{x}}{dv}}{\|\frac{d\vec{x}}{du} \times \frac{\delta\vec{x}}{dv}\|}}$

Example: Torus $\vec{x} = \langle[2+\cos(v)]\cos(u), [2+\cos(v)]\sin(u), \sin(v)\rangle, u, v \in [0, 2\pi)$. What's the tangent plane at $u = \frac{\pi}{4}, v = 0$?

$$d\vec{x}/du = \langle -\sin(u)(2+\cos(v)), \cos(u)(2+\cos(v)), 0\rangle \tag{34}$$
$$d\vec{x}/dv = \langle -\sin(v)\cos(u), -\sin(v)\sin(u), \cos(v)\rangle \tag{35}$$
$$u = \frac{\pi}{4}, v = 0: \tag{36}$$
$$d\vec{x}/du = \langle -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0, \rangle \tag{37}$$
$$d\vec{x}/dv = \langle 0, 0, 1\rangle \tag{38}$$
$$dx/du \times dx/dv = \langle \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0\rangle \tag{39}$$
$$\hat{n} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle \tag{40}$$
$$\hat{n} \cdot \vec{x} = 0 \rightarrow \hat{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \tag{41}$$
$$\rightarrow ... \rightarrow x + y = 3\sqrt{2} \tag{42}$$
$$\tag{43}$$

5.1 Example: Ellipsoid $x^2+2y^2+z^2 = 4$ What's the normal at $(1, \frac{1}{\sqrt{2}}, \sqrt{2})$?

Method 1: parametrize with spherical u, v First, transform to sphere with change of coordinates, then flip to speherical coordinates.

$$x^2 + 2y^2 + z^2 = 4 \tag{44}$$
$$X = x/2, Y = \frac{Y}{\sqrt{2}}, Z = z/2 \tag{45}$$
$$X^2 + Y^2 + Z^2 = 1 \tag{46}$$
$$X = \cos(u)\sin(v), Y = \sin(u)\sin(v), Z = \cos(v) \tag{47}$$
$$p = (1, \frac{1}{\sqrt{2}}, \sqrt{2}) \rightarrow u = v = \frac{\pi}{4} \tag{48}$$
$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle -1, \frac{1}{\sqrt{2}}, 0\rangle \tag{49}$$
$$\frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \frac{1}{\sqrt{2}}, -\sqrt{2}\rangle \tag{50}$$
$$\frac{dx}{du}(\frac{\pi}{4}, \frac{\pi}{4}) \times \frac{dx}{dv}(\frac{\pi}{4}, \frac{\pi}{4}) = \langle 1, \sqrt{2}, \sqrt{2}\rangle \tag{51}$$
$$\hat{n}_{out} = \frac{\langle -1, -\sqrt{2}, -\sqrt{2}\rangle}{\sqrt{5}} \tag{52}$$

Method 2: rewrite as z = g(x,y)

$$
\begin{align}
x^2 + 2y^2 + z^2 = 4 \tag{53}\\
z = (4 - x^2 - 2y^2)^{\frac{1}{2}} \tag{54}\\
dz/dx = \frac{1}{2} \times -2x(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -\frac{1}{\sqrt{2}} \tag{55}\\
dz/dy = \frac{1}{2} \times -4y(4 - x^2 - 2y^2)^{-\frac{1}{2}} = -2\sqrt{2}/\sqrt{2} = -1 \tag{56}\\
f \approx \sqrt{2} + dz/dx(1, \frac{1}{\sqrt{2}})(x-1) + dz/dy(1, \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) \tag{57}\\
\to ... \to \frac{1}{\sqrt{2}}x + y + z = 2\sqrt{2} \tag{58}\\
\tag{59}
\end{align}
$$

giving us normal vector $\langle \frac{1}{\sqrt{2}}, 1, 1\rangle = \frac{\langle 1, \sqrt{2}, \sqrt{2}\rangle}{\sqrt{5}}$ after normalization.

Method 3: gradient

Gradient is always normal to the tangent plane. Recognize level set of $f(x,y,z) = x^2 + 2y^2 + z^2$.

$\nabla f = \langle 2x, 4y, 2z\rangle \to \nabla f(1, \frac{1}{\sqrt{2}}, \sqrt{2}) = \langle 2, 2\sqrt{2}, 2\sqrt{2}\rangle$

Then normalize.

5.2 Mobius strip and "outward direction"

Mobius strip is

- $x = 2\cos(u) + v\cos(\frac{u}{2})$
- $y = 2\sin(u) + v\cos(\frac{u}{2})$
- $z = v\sin(\frac{u}{2})$
- $u \in [0, 2\pi], v \in [-\frac{1}{2}, \frac{1}{2}]$

$\hat{n} = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$ at (0,0) is $\langle 0, 0, -1\rangle$,

but at the same point $(2\pi, 0)$ $\hat{n} = \langle 0, 0, 1\rangle$!!

6 Chapter 2.6: Vector Fields

(Lots of intuition questions here...)

One nugget: using **gradient vector fields**: Suppose $\vec{F}(x,y) = \langle 2, -4y^3 \rangle$. If $\vec{F} = \nabla f$ for some (single value function) f, then F's arrows are perpendicular to a level set $f = c$. So look at $f = 2x - y^4$ and find perpendicular arrows to these. That's actually F!

Linear approximation for $\vec{F} : D \in \mathbf{R}^n \to \mathbf{R}^m$

Main idea: $\vec{F}(\vec{x}) = \vec{F}(\vec{a}) + A(\vec{a})(\vec{x} - \vec{a})$

Note that A takes in vectors of size n (so it has as many columns as the input space), and has m functions (rows) that operate on it. So the Jacobian, A, has as row i, column j, the quantity $\frac{dF_i}{dx_j}(\vec{a})$.

$dF_i/d\vec{x}$ extends across row i.

7 Chapter 2.7: Jack and the Beanstalk (Newton's method)

Basis for Newton's:

If we're estimating x_1 by following the derivative at x_0, this means we're looking at the line with x-intercept x_1, with slope $f'(x_0)$.

So instead of $y = mx + b$, we'll flip the two and use

$x = y/m + x_{int}$

or $x_0 = f(x_0)\frac{1}{f'(x_0)} + x_1$,

or $\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$

Note that, under Newton's something like $|x|$ will converge immediately, x^3 will converge moderately, and a S-curve might barely converge if at all.

The extension of this with the Jacobian matrix $A = DF'(x_0)$ is $\vec{x}_1 = \vec{x}_0 - (D\vec{F}(\vec{x}_0))^{-1}\vec{F}(x_0)$

8 Chapter 2.8: Electrostatic bootcamp

Electric charge radiates out equally in all directions, and is inversely proportional to distance.

Formula, with Q as the charge, ϵ_0 is a constant: $\vec{E}(x,y,z) = \frac{Q}{4\pi\epsilon_0\|x\|^2}\hat{x}$

A field line is a special case of a **flow line** - the space curve that follows $\vec{F}$'s arrows. The tangent vector to the flow line is $\vec{F}(\vec{x}(\tilde{t}))$ ($\tilde{t}$ is not time here), so $\frac{d\vec{x}}{d\tilde{t}} = \vec{F}(\vec{x}(\tilde{t}))$

Example: Vector field $\vec{F}(x,y) = \langle -2y, 3x \rangle$. What's the flow line through (2,0)?

Solution: Need to solve $dx/dt = -2y, dy/dt = 3x$. Key is "separating the equations". Remember x and y are functions of t!

$$\frac{d^2x}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6x. \tag{60}$$
$$\frac{d^2y}{dt^2} = -2\frac{dy}{dt} = -2 \times 3x = -6y. \tag{61}$$
$$x(t) = -6x''(t), y(t) = -6y''(t) \tag{62}$$
$$\rightarrow x = A\cos(\sqrt{6}t) + B\sin(\sqrt{6}t), y = C\cos(\sqrt{6}t) + D\sin(\sqrt{6}t) \tag{63}$$
$$\frac{dx}{dt} = -2y(t) \rightarrow \frac{\sqrt{6}}{2}A\sin(\sqrt{6}t) - \frac{\sqrt{6}}{2}B\cos(\sqrt{6}t) = y(t) \tag{64}$$
$$x(t=0) = 2 \rightarrow A = 2 \tag{65}$$
$$y(t=0) = 0 \rightarrow B = 0 \tag{66}$$
$$\vec{F}(t) = \langle 2\cos(\sqrt{6}t), \sqrt{6}\sin(\sqrt{6}t)\rangle \tag{67}$$
$$\tag{68}$$

Note: **Field lines** follow rules:

- Go from positive charges to negative
- Density of lines directly relates to how much charge a point has
- Lines don't intersect.
- Corollary: If count of out equals count of in, point has zero charge
- "Number" (to be defined) of field lines in and out of a *surface* related to the charge inside. Upcoming.

9 Chapter 3: Surface integrals, Flux, Divergence

9.1 3.1: Surface Integrals

Example: Fluid pressure in a tank is:

- Proportional (via some weight constant p_{fluid}) to depth of the point
- Pushes out via the normal $\hat{n}$
- So, for the $x = l$ side of a cube of length l, this would be

  $\vec{F}_{x=l} = (\iint_{[0,l]\times[0,l]} p_{fluid}[1 - \frac{z}{l}]dydz)\hat{\imath}$

Example: Hemisphere of size l, sitting at (0, 0, 0)

Finding the out pointing unit normal of hemisphere at point $(x, y, \sqrt{l^2 - x^2 - y^2})$

Note: Can just eyeball this, but one way is the **gradient**.

First, the relation is $x^2 + y^2 + (z-l)^2 = l^2$. Make it a function g and take the level set at l^2:

$$g(x,y,z) = x^2 + y^2 + (z-l)^2 = l^2 \tag{69}$$

$$\nabla g(x,y,z) = \langle 2x, 2y, 2(z-l)\rangle \tag{70}$$

$$\hat{n} = \pm\frac{\nabla g(x,y,z)}{\|\nabla g(x,y,z)\|} \tag{71}$$

$$\hat{n} = \pm\frac{\langle x, y, (z-l)\rangle}{\sqrt{x^2 + y^2 + (z-l)^2}} \tag{72}$$

$$\hat{n} = \pm\frac{\langle x, y, (z-l)\rangle}{\sqrt{l^2}} \tag{73}$$

$$\hat{n} = \pm\langle \frac{x}{l}, \frac{y}{l}, \frac{z}{l} - 1\rangle \tag{74}$$

$$\tag{75}$$

Note: Integrating over a patch dA on the surface means finding the area of micro-patches ΔA_{ij}, which is the parallelogram defined by

$$s_1 = \langle \Delta x_i, 0, \Delta x_i f_x(x_i^*, y_j^*)\rangle \tag{76}$$

$$s_2 = \langle 0, \Delta y_j, \Delta y_j f_y(x_i^*, y_j^*)\rangle \tag{77}$$

$$\Delta A_{ij} \approx \|s_1 \times s_2\| \tag{78}$$

$$= \sqrt{(1 + [f_x(x_i^*, y_j^*)]^2 + [f_y(x_i^*, y_j^*)]^2}\Delta x_i \Delta y_j \tag{79}$$

$$\tag{80}$$

So if $\boxed{z = f(x,y), dA = \sqrt{1 + f_x^2 + f_y^2}}$.

So the total pressure ends up being $\vec{F}_{tot} = p_{fluid} \iint (p \cdot \hat{n}) dA$

$$= p_{fluid} \iint_{x^2+y^2 \leq l^2} [1 - \frac{f(x,y)}{l}] \hat{n} \sqrt{1 + [f_x]^2 + [f_y]^2} dxdy \tag{81}$$

$$f(x,y) = l - \sqrt{l^2 - x^2 - y^2} \tag{82}$$

$$\hat{n} = \langle \frac{x}{l}, \frac{t}{l}, \frac{f(x,y)}{l} - 1 \rangle \tag{83}$$

$$f_x = \frac{x}{\sqrt{l^2 - x^2 - y^2}}, f_y = \frac{y}{\sqrt{l^2 - x^2 - y^2}} \tag{84}$$

And for the only-nonzero component, $\hat{k}$, this simplifies after a lot of hand-math to $F_{tot} = -p_{fluid} (\iint_{x^2+y^2 \leq l^2} \sqrt{1 - (\frac{x^2+y^2}{l^2})} dxdy) \hat{k}$

Side Note during solving: $dxdy \to rdrd\theta$.

- TODO: This looks to be something to do with the determinant of the Jacobian matrix F_i/x_j.
- Intuitively, consider that a patch $dx \cdot dy$ is a slice of a big disk which has dimensions dr on the ray, $rd\theta$ on the arc.

10 3.2: Flux Part I

Main idea: Field lines are innumerable, so counting them through a surface makes no sense. Instead, we'll use **flux** to help us measure charge pushed through a surface per unit time.

Example: If charge q of mass m in a field of $\vec{E} = E_0 \hat{i}$ moves from origin along x towards R according to $\frac{d^2x}{dt^2} = \frac{q}{m} E_0$, then solving the diff eq. means that $x = \frac{q}{2m} E_0 (\Delta t)^2 = R$. This means we're pushing all charges within $\frac{q}{2m} E_0 (\Delta t)^2$ to the left of the disk through it.

Then, if we're considering a cylinder of base area A, mass density δ, charge density ρ:

- Every test charge chunk ΔV within $\frac{\rho \Delta V}{2\delta \Delta V} E_0 (\Delta t)^2$ passes through. That's the height.
- Area is A, so total volume is $\frac{\rho (\Delta t)^2}{2\delta} E_0 A$
- Density of charge per volume is ρ, so total is $\frac{\rho^2 (\Delta t)^2}{2\delta} E_0 A$

Note: Tilting this forward from the z-axis by θ multiplies the cross-section area of the cylinder (now an ellipse) by $\cos(\theta)$. Can work out the ellipse volume, or just note that each "Riemann bar" orthogonal to x-axis just got squished by $\cos(\theta)$.

So we define **flux** as amount of charge through a closed surface. $\boxed{\Phi = (\vec{E} \cdot \hat{n})A}$ if $\vec{E}$ is a constant field. (Units: joules/second/m^2, or watts/m^2), and $\boxed{\Phi = \iint_S (\vec{E} \cdot \hat{n})dA}$ generally.

We can further note $(\vec{E} \cdot \hat{n}) = \|\vec{E}\| \cos(\theta)$ by last problem.

Example: Flux through an empty cube from the origin is necessarily 0 since every face cancels the other.

Another example: A square pyramind with top at $(0,0,1)$, sides at 1 on each axis:

- All the triangles will cancel in the x, y directions.
- A triangle $(1,0,0)(0,1,0),(0,0,1)$ has two displacement vectors $P_1P_3 = P_3 - P_1 = (-1,0,1), P_2P_3 = (0,-1,1)$.
- $P_1P_3 \times P_2P_3 = (1,1,1) \rightarrow \hat{n} = \frac{(1,1,1)}{\sqrt{3}}$
- $A = \frac{1}{2}\|P_1P_3 \times P_2P_3\| = \frac{\sqrt{3}}{2}$
- $\Phi = (\vec{E} \cdot \hat{n})A = (E_0\frac{1}{\sqrt{3}})\frac{\sqrt{3}}{2} = \frac{E_0}{2}$
- So total flux through these is $4 \cdot \frac{1}{2}E_0 = 2E_0$
- However, the bottom has area $\sqrt{2}^2 = 2$ and flux E_0 , so total is 0!

11 3.3: Flux Part II

Note:

- Charge (q) is the volts of the point charge. Total charge Q_{tot} is total charge inside some surface.
- Electric field is sum of those point charges acting at a distance, and a is a single vector.
- Flux is the sum of the electric field flowing through a surface.
- Total charge Q_{tot} of a surface is basically the sum of all the flux going in/out, except that it's that divided by some constant ϵ_0.

Note: $\vec{E}$ isn't usually constant, and the surface S is usually curved. So we need calculus to break up surface S into small pieces ΔA_i and evaluate $\vec{E}_i$ there at that normal $\hat{n}_i$. So

$\sum_{patches}(\vec{E}_i \cdot \hat{n}_i)\Delta A_i = \iint_S (\vec{E} \cdot \hat{n}))dA = \Phi$

Easy Example: If, say, $(\vec{E} \cdot \hat{n}) = 1$ everywhere, we're just looking at $\iint_S dA$, or the total surface area.

Another example. Given:

- Real electric field law: $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3}$
- Real observation: Total electric flux through a surface (Φ) is proportional to total charge inside (Q_{tot}). $\Phi = \iint_S (\vec{E} \cdot \hat{n}) \Delta A \propto Q_{tot}$
- Then constant must be $\frac{1}{\epsilon_0}$. Why?
  - On unit sphere, $\hat{n} = \frac{\vec{x}}{\|\vec{x}\|}$
  - So $\vec{E} \cdot \hat{n} = \frac{q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3} \cdot \hat{n}$
  - $= \frac{q}{4\pi\epsilon_0}$ since $\|\vec{x}\| = 1$ on unit sphere
  - Then $\Phi = \iint_S \frac{q}{4\pi\epsilon_0} dA$
  - $= \frac{q}{4\pi\epsilon_0} 4\pi$ by surface area of unit sphere
  - $= \frac{q}{\epsilon_0}$
- Therefore, because all of the field goes through the surface (no matter the shape), **Gauss's law** says $\iint_S (\vec{E} \cdot \vec{n}) dA = \frac{Q_{tot}}{\epsilon_0}$

Note: Because (UNEXPLAINED!) symmetry of a contained *ball* implies that, for distance ρ from origin, $\vec{E} = E(\rho)\hat{\rho}$, the above works the same for a point charge or a uniform (contained) ball.

Example: For a big radius R ball of charge Q containing a small ball of radius ρ with charge Q_{tot}, what must the charge $E(\rho)$ at any point be?

- Small charge Q_{tot} is proportional to volume of the big charge Q by $Q_{tot} = Q\frac{V_{small}}{V_{big}} = Q\frac{\rho^3}{R^3}$
- $\frac{Q_{tot}}{\epsilon_0}$ = total charge $= \iint_S E(\rho)(\|\hat{\rho}\|) dA = E(\rho) \iint_S 1 dA = E(\rho) 4\pi\rho^2$
- So $\frac{Q_{tot}}{\epsilon_0} = Q\frac{\rho^3}{R^3\epsilon_0} = E(\rho)4\pi\rho^2$
- So $E(\rho) = \frac{Q}{4\pi\epsilon_0} \frac{\rho}{R^3}$

Example: Infinite wire, x=y=0, charge per length is λ. What's the magnitude of the field r units away?

- Use a cylinder.

- What's the total charge of the cylinder? Top and bottom are perpendicular to the field so can be ignored.
- There's some function $E(r)$ which, time $\hat{r}$, is the field by symmetry.
- $\Phi = \iint_{cylinder} (E(r) \cdot \hat{r}) dA = E(r) \iint_{cylinder} 1 dA = E(r) 2\pi r h.$
- $\frac{Q_{tot}}{\epsilon_0} = E(r) 2\pi r h \Rightarrow E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$

Example: Infinite plane, x=y=0, charge per area is σ. What's the mangitude of the field at height h?

- Use a cylinder again
- What's the total charge of the cylinder? Side is perpendicular to the field so can be ignored. Looking at top and bottom, $\phi = 2EA + 2EA.$, where E is charge through the top.
- $2EA = \frac{\sigma A}{\epsilon_0} \rightarrow E = \frac{\sigma}{2\epsilon_0}$
- Note: It appears it's height-invariant!

12 3.4: Surface Integrals

- Flux is a specific form of the general $\iint_S F da$.
- dA is a patch of a parallelogram on the surface. This is defined by corners $\vec{x}(u_0, v_0), \vec{x}(u_0, v_0) + \delta_u \vec{x}(u_0, v_0)$, and $\vec{x}(u_0, v_0) + \delta_v \vec{x}(u_0, v_0)$
- Therefore, using the parallelogram area formula,$dA = \Delta_u \Delta_v \|\vec{x}_u \times \vec{x}_v\|$
- Taking to the limit, this means the area is $\iint_D F(\vec{x}(u, v)) \|\vec{x}_u \times \vec{x}_v\| dudv$

Example: Sphere $x^2 + y^2 + z^2 = R^2$ surface area. Take θ as angle around ϕ as angle from top of z axis.

- Parametrization $x = R \sin\phi \cos\theta, y = R \sin\phi \sin\theta, z = R \cos\phi$
- $dx/d_\theta = -R \sin\phi \sin\theta, dy/d_\theta = R \sin\phi \cos\theta, dz/d_\theta = 0$
- $dx/d_\phi = R \cos\phi \cos\theta, dy/d_\phi = R \cos\phi \sin\theta, dz/d_\phi = -R \sin\phi$
- After working it out, $dx/d_\theta \times dx/d_\phi = R^2 \sin\phi \langle -\sin\phi \cos\theta, -\sin\phi \sin\theta, -\cos\phi \rangle$
- Doing the math, $\|dx/d_\theta \times dx/d_\phi\| = R^2 \sin\phi$
- So $\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} 1 \cdot R^2 \sin\phi = 2\pi \int_{\phi=0} \pi R^2 \sin\phi = 2\pi R^2 [-\cos\phi]_0^\pi = 4\pi R^2$

Example: Parabaloid $z = 1 - x^2 - y^2, x^2 + y^2 \leq 1$

- Parametrization $x = R\sin\phi\cos\theta, y = R\sin\phi\sin\theta, z = R\cos\phi$
- $dz/dx = \langle 1, 0, -2x\rangle, dz/dy = \langle 0, -1, -2y\rangle$
- $\|dz/dx \times dz/dy\| = 1 + 4x^2 + 4y^2$
- Area $= \iint_D 1 \cdot dA = \iint_D \sqrt{1 + 4x^2 + 4y^2}dxdy$
- Change to polar, remembering this square depends on r: $\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\sqrt{1 + 4r^2\cos^2\pi 4r^2\sin^2\pi}rdrd\theta = 2\pi\int_0^1\sqrt{1 + 4r^2}rdr$
- After working it out, this ends up being $[\frac{2}{3}\cdot\frac{1}{8}(4r^2+1)^{\frac{3}{2}}]_0^1 = \frac{\pi}{6}(5\sqrt{5}-1)$

Example: Torus $x(u, v) = [R+r\cos(u)]\sin(v)], y(u, v) = [R+r\cos(u)]\cos(v)], z = r\sin(u), u, v \in [0, 2\pi)$

- Already parametrized in polar, basically,
- $d\vec{x}/du = \langle -r\sin(u)\sin(v), -r\sin(u)\cos(v), r\cos(u)\rangle$
- $d\vec{x}/dv = \langle Rcos(v) + r\cos(u)\cos(v), -R\sin(v) - r\cos(v)\sin(v), 0\rangle$
- After lots of math, $\|d\vec{x}/du \times \vec{x}/dv\| = r(R + r\cos(u))$
- $\int_{u=0}^{2\pi}\int_{v=0}^{2\pi} r(R + r\cos(u)du = 2\pi r\int_{u=0}^{2\pi} r(R + r\cos(u))du$
- $= 2\pi r[2\pi R] = 4\pi^2 Rr$

Example: Center of mass of unit (hollow?) hemisphere sitting on origin.

- Center of mass for density ρ is $\frac{\iint_S \vec{x}\rho dA}{\iint_S \rho dA}$
- Obvious that x, y center at zero.
- For denominator, $\iint_S dA$ is just surface area, or half of $4\pi 1^2 = 2\pi$.
- For numerator:
  - Do typical θ, ϕ parametrization.
  - $\vec{x}_\theta \times \vec{x}_\phi = \langle\sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi)\rangle$
  - Pull out the $\sin(\phi)$ and the remaining norm is one, so $\|\vec{x}_\theta \times \vec{x}_\phi\| = \sin(\phi)$
  - $\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} z \cdot dA = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\cos(\phi)\sin(\phi) = \frac{1}{2}$

Example: Moment of inertia

- Formula: $I_z = M\iint_S(x^2 + y^2)dA$.
- Object to spin: helicoid $\vec{x}(\theta, v) = \langle\theta\cos(v), \theta\sin(v), v\rangle\theta \in [0, R], v \in [0, 2\pi]$

- Assumption for the problem: $\int_{\theta=0}^{\theta=R} \theta^2\sqrt{1+\theta^2}d\theta = 2$
- Center of mass for density ρ is $\frac{\iint_S \vec{x}\rho dA}{\iint_S \rho dA}$
- Use polar coordinates r, θ.
- After computation, $\|\vec{x}_r \times \vec{x}_\theta\| = \sqrt{1+r^2}$
- $M\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi}\sqrt{1+r^2}(r^2\cos^2(\theta)+r^2\cos^2(\theta))dA = M\iint\sqrt{1+r^2}r^2 = 2\pi M\iint\sqrt{1+r^2}r^2 = 4\pi$ by hint

Example: Flux through unit hemisphere

- Formula: $\Phi = \iint_S(\vec{E}\cdot\vec{n})dA = \iint_S FdA$
- Field: $\vec{E} = \langle yz, xz, xy\rangle$
- Use polar coordinates
- **Base**: $\hat{n} = -\hat{k}$ so $\langle yz, xz, xy\rangle\cdot\langle 0,0,-1\rangle = -xy$ It's clear by symmetry that $\iint_{u^2+v^2\leq 1} -xydxdy = 0$
- **Top**: Set $u = \theta \in [0, 2\pi), v = \phi \in [0, \frac{\pi}{2}]$.
- As usual, $dA = \|\vec{x}_u \times \vec{x}_v\| = \sin(v)$.
- Norm just points out from the center: $\hat{n} = \langle\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle$
- $\vec{E} = \langle\sin(u)\sin(v)\cos(v), \cos(u)\sin(v)\cos(v), \cos(u)\sin^2(v)\sin(u)\rangle$
- So $\vec{E}\cdot\hat{n} = 3\cos(u)\sin(u)\cos(v)\sin^2(v)$
- Looking at this, this is really $\int_{u=0}^{u=2\pi} k(v)\sin^2(u)$ for some $k(v)$, so this will be 0.
- Therefore, total flux is zero, and by Gauss's law, total field contained inside has to be 0 too.

Example: Field $\vec{E} = \ln(x^2+y^2)\langle x, y, 0\rangle$ through R-wide cylinder, height h

- Parameterize: $x = r\cos\theta, y = r\sin\theta, z = z$
- **Top/Bottom**: $\hat{n} = \langle 0,0,1\rangle, \vec{E} = f(x,y)\langle x,y,0\rangle \to \hat{n}\cdot\vec{E} = 0$
- **Side**: $\hat{n} = \frac{1}{R}\langle R\cos(\theta), R\sin(\theta), 0\rangle$
- $\Phi = \iint_{cylinder}\frac{1}{R}\langle R\cos(\theta), R\sin(\theta), 0\rangle\cdot\langle R\cos(\theta), R\sin(\theta), 0\rangle\ln(R^2\cos^2(\theta)+R^2\sin^2(\theta))$
- $= R\iint_{cylinder}\ln(R^2)+\ln(\cos^2(\theta)+\sin^2(\theta)) = R\cdot 2\ln(R)\cdot h\cdot 2\pi R = 4\pi R^2\ln(R)h$

Example: Field $\vec{E} = e^{-x^2-y^2-z^2}\vec{x}$ with sphere S at radius R, setting $\epsilon_0 = 1$

- Parameterize: $x = R\cos(\theta)\sin(\phi), y = R\sin(\theta)\sin(\phi), z = R\cos(phi)$
- $\hat{n} = \langle \cos(\theta)\sin(\phi), y = \sin(\theta)\sin(\phi), z = \cos(phi)\rangle$
- $\vec{x} = R\hat{n}$, so $\vec{E}\cdot n = Re^{-R^2}$
- $R\iint_{sphere} e^{-R^2} = 4\pi R^3 e^{-R^2}$

Note: In the future we write $\boxed{\hat{n}dA = \vec{dA}}$

12.1 3.5: Divergence part I

Main idea: Last chapter was all about having field $\vec{E}$ and wanting to figure out Q_{tot} (or $\frac{\phi}{\epsilon_0}$). Usually, we have the charge distribution Q and want to figure out $\vec{E}$. Most of the field derivation from 3.3 was through tricks for highly symmetric spaces (infinite line, infinite plane, uniform ball, etc.)

Point: The flux through a sphere in a uniform field is zero. Why? Move the center point to the origin, rotate so field is $\hat{k}$ (both don't change the flux), and consider that what goes out at $\langle x, y, z\rangle$ comes in at $\langle x, y, -z\rangle$. This same argument applies for $\iint_{S=sphere} \hat{n}_i\hat{n}_j dA$, where i, j are components in $\{x, y, z\}$.

However, if $i = j$, then $\iint_S \hat{n}_i\hat{n}_j dA = \iint_S \hat{n}_i^2 = \frac{4}{3}\pi R^2$, since $\iint_S(\hat{n}_x^2 + \hat{n}_y^2 + \hat{n}_z^2)dA = \iint_S 1dA = 4\pi R^2$, so each of the components must be a third of that.

12.1.1 Defining Divergence

Remember that, in Gauss's law $\frac{Q}{\epsilon_0} = \iint_S \vec{E}\cdot\vec{dA}$, we're using information about $\vec{E}$ spread out over surface S. We can also shrink this to a smaller surface.

Shrinking to a point $\vec{P}$, $\lim_{R\to 0}\frac{1}{4\pi R^3}\iint_S \vec{E}\cdot\vec{dA} = \frac{Q_{tot}}{\epsilon_0 4\pi R^3} = \frac{\rho(\vec{P})}{\epsilon_0}$. (This works by dividing both sides by volume of a sphere)

Deriving Divergence: Computing $\lim_{R\to 0}\frac{1}{4\pi R^3}\iint_S \vec{E}\cdot\vec{dA}$

- $\iint_S \hat{n}_i\hat{n}_j dA = 0$ if $i \neq j$
- $\iint_S \hat{n}_i\hat{n}_j dA = \frac{4}{3}\pi R^3$ if $i = j$
- Use linear approximation with Jacobian $D = \frac{\delta E_i}{\delta x_j}$, $\vec{E}(\vec{x}) = \vec{E}(\vec{P}) + D\vec{E}(\vec{P})(\vec{x} - \vec{P})$
- $\iint_S \vec{E}(\vec{P}) = 0$ for any constant. (think of the flux of a sphere in a constant field as above)
- This leaves $D\vec{E}(\vec{P})(\vec{x}-\vec{P})\cdot\hat{n} = \sum_{i,j}\hat{n}_i[\vec{x}-\vec{P}]_j D\vec{E}(\vec{P})_{ij}$
- Since it's a sphere, the normal $\hat{n} = \frac{\vec{x}-\vec{P}}{R}$

- Therefore $D\vec{E}(\vec{P})(\vec{x}-\vec{P})\cdot\hat{n} = R\sum \hat{n}_i\hat{n}_j D\vec{E}(\vec{P})_{ij}$ (swap $R\hat{n}_j$ for $[\vec{x}-\vec{P}]_j$)
- These terms are all 0 except where $i=j$, so $D\vec{E}(\vec{P})(\vec{x}-\vec{P})\cdot\hat{n} = \frac{4}{3}\pi R^2 \times R \times [\frac{\delta E_x}{\delta x} + \frac{\delta E_y}{\delta y} + \frac{\delta E_z}{\delta z}]$
- This equals $\lim_{R\to 0}\frac{1}{4\pi R^3}\iint_S \vec{E}\cdot d\vec{A}$ so eliminating the sphere volume gives us $\boxed{\frac{\rho(\vec{P})}{\epsilon_0} = [\frac{\delta E_x}{\delta x} + \frac{\delta E_y}{\delta y} + \frac{\delta E_z}{\delta z}] = \nabla\cdot\vec{E}}$

We can think of the divergence ∇, also like an operator:
$\boxed{\nabla\cdot\vec{F} = \nabla\cdot(F_x\hat{i} + F_x\hat{j} + F_x\hat{k}) = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})\cdot(F_x\hat{i} + F_x\hat{j} + F_x\hat{k})}$

Shifting to Cylindrical Coordinates: If instead we want to describe $\vec{F} = \vec{F}_r\hat{r} + \vec{F}_\theta\hat{\theta} + \vec{F}_z\hat{z}$, we have $\boxed{\nabla\cdot\vec{F} = \frac{1}{r}\frac{\delta rF_R}{\delta r} + \frac{1}{r}\frac{\delta F_\theta}{\delta\theta} + F_z\frac{\delta F_z}{\delta z}}$. How to derive?

- Note identities $\hat{r} = \cos(\theta)\hat{i} + \sin(\theta)\hat{j}, \hat{\theta} = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j}$. If $\theta = 0$, these point right and up, corresponding to $\hat{i},\hat{j}$. If θ rotates, these do too.
- $F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = \vec{F} = (F_r(\cos(\theta)\hat{i} + \sin(\theta)\hat{j}) + F_\theta(-\sin(\theta)\hat{i} + \cos(\theta)\hat{j}) + F_z\hat{k}$
- Rearrange so that $\vec{F} = F_x\hat{i} + F_x\hat{j} + F_x\hat{k} = (F_r\cos(\theta) + F_\theta(-\sin(\theta))\hat{i} + (F_r\sin(\theta) + F_\theta\cos(\theta))\hat{j} + F_z\hat{k}$.
- Compute $\frac{\delta}{\delta x} = \frac{\delta}{\delta r}\frac{\delta r}{\delta x} + \frac{\delta}{\delta\theta}\frac{\delta\theta}{\delta x} = \frac{\delta}{\delta x} = \cos(\theta)\frac{\delta}{\delta r} - \frac{\sin(\theta)}{r}\frac{\delta}{\delta\theta}$.
  - The second term: $\frac{d\theta}{dx} = \tan^{-1}(y/x)) = \frac{y}{1+y^2/x^2} * \frac{-1}{x^2} = -\frac{r\sin(\theta)}{r^2(\sin^2+\cos^2)} = -\frac{\sin(\theta)}{r}$
- Do something similar for similar for $\frac{d}{dy}$ in the second term.
- Combine and shake it out.

Shifting to Spherical Coordinates: Using a similar process, we get
$\boxed{\nabla\cdot\vec{F} = \frac{1}{\rho^2}\frac{\delta(\rho^2F_\rho)}{\delta\rho} + \frac{1}{\rho\sin(\phi)}\frac{\delta}{\delta\phi}(\sin(\phi)F_\phi) + \frac{1}{\rho\sin(\phi)}\frac{\delta F_\theta}{\delta\theta}}$

12.2 3.6: Divergence Part 2

Example: Compute divergence of electric field $E = \frac{Q}{4\pi\epsilon_0}\frac{\vec{x}}{\|\vec{x}\|^3}$ outside radius R.

- $\frac{\delta E_x}{\delta x}(\frac{Q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}) = v\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$.
- Symmetrical for $\frac{\delta E_y}{\delta y}, \frac{\delta E_z}{\delta z}$
- Sums to 0.

Example: Compute divergence of electric field $E = \frac{Q}{4\pi\epsilon_0}\frac{\vec{x}}{R^3}$ inside radius R.

- $\frac{\delta E_x}{\delta x}(\frac{Q}{4\pi\epsilon_0}\frac{x}{R^3}) = \frac{Q}{4\pi\epsilon_0}\frac{1}{R^3}$
- Symmetrical for $\frac{\delta E_y}{\delta y}, \frac{\delta E_z}{\delta z}$
- Sums to $\frac{Q}{4\pi\epsilon_0}\frac{3}{R^3}$

So, the divergence of an electric field is proportional to $\frac{Q}{R^3}$ inside the sphere, and 0 outside the sphere.

So, divergence at a point intuitively measures **how much the field spreads out** or sinks into the point. For electric charge, $\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$ means that at that point, the spready-ness is proportional to the charge.

Example: if $\epsilon_0 = 1$ and the field is $\vec{E} = x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$, how much charge is in the $[0,1] \times [0,1] \times [0,1]$ box?

- Answer: $\rho = \nabla \cdot \vec{E} = 1 + 2 + 1 = 4$. So 4 units.

Another Example: if $\vec{E} = \sin(yz)\hat{\imath} + \sin(xz)\hat{\jmath} + \sin(xy)\hat{k}$ in some complicated surface, then what?

- Noticing that $\nabla \cdot \vec{E} = 0$ shows you this is 0 no matter the shape of the region. This means *the vectors pointing into the region (in fact, any part of the space) are balanced out by those pointing out from the region.*

12.3 3.7: The Divergence Theorem

The Divergence Theorem falls out of equating finding charge Q with a double integral over a bounded surface with the triple integral of the contained volume:

- $\frac{Q}{\epsilon_0} = \iint_S \vec{E} \cdot d\vec{A}$ (Gauss's law)
- $\Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ within R (Proved Divergence equivalent from last section)
- $Q = \iiint_R \rho dxdydz$ (Just integrating charge over volume)
- $\Rightarrow Q = \iiint_R \rho dxdydz = \epsilon_0 \iiint_R \nabla \cdot \vec{E} dxdydz = \epsilon_0 \iint_S \vec{E} \cdot d\vec{A}$
- $\boxed{\Rightarrow \iint_S \vec{E} \cdot d\vec{A} = \iiint_R \nabla \cdot \vec{E} dxdydz}$ (Divergence Theorem). Integrating the field over the bounding surface is the same as integrating the divergence over the volume.

Smooshy thought: This looks like another version of Fundamental Theorem of Calculus. The integral of the function evaluated at the boundaries is the same as the function summed inside the boundary.

Proving Divergence Generally: We're gluing micro-cubes together and not changing the total flux. This means any surface is the flux going in and out of its "skin".

- Note that since the flux outward through a cube face is the negative of it inward, gluing two cubes together on this face means we're summing the total fluxes.
- Do this for tiny cubes approximating the surface we care about.
- In a cube centered on point $P, F \approx \vec{F}(P) + D\vec{F}(P)(\vec{x} - \vec{P})$.
- $\iint_S \vec{F}(P)d\vec{A} = 0$ since it's constant, since every face i has normal $\hat{n}_i$, and a partner of equal size with normal $-\hat{n}_i$.
- However, for a cube of side ϵ the flux through, say, Face I ($x = \epsilon + P$) is $\iint_S D\vec{F}(P)(\vec{x} - \vec{P})$ ends up being $\frac{\delta F_x}{\delta x} 4\epsilon^3$, since:
  - Consider side $x = p_x + \epsilon$
  - $D\vec{F}(P)(\vec{x} - \vec{P}) \cdot \hat{n} = [D\vec{F}(P)]_{xx}(x - p_x) + [D\vec{F}(P)]_{xy}(x - p_y) + [D\vec{F}(P)]_{xz}(z - p_z)$.
  - So, the functons that consider the inputs of y, z don't matter.
  - So $\iint_{FaceI}(y - p_y)dA = 0$ around p_y by symmetry. Same for z on that face.
  - But for x, $\iint_{FaceI}(x - p_x)dA = \int_{p_y-\epsilon}^{p_y+\epsilon} \int_{p_z-\epsilon}^{p_z+\epsilon} \epsilon dydz = 4\epsilon^3$
  - $D_{ij}\vec{F}(P)$ is constant for all $i, j \in \{x, y, z\}$, so this face is then $\frac{\delta F_x}{\delta x} 4\epsilon^3$.
  - Summing the opposite face (with the same flux), yields $\frac{\delta F_x}{\delta x} 8\epsilon^3 = \frac{\delta F_x}{\delta x} V$.
  - Summing across the other faces yields $\frac{\delta F_x}{\delta x} V + \frac{\delta F_y}{\delta y} V + \frac{\delta F_z}{\delta z} V$.

  Finally, this shows the **flux on one of these microcubes is** $\boxed{\nabla \cdot \vec{F}(P)V.}$

  In total, the **divergence theorem**: $\boxed{\iint_{\delta C} \vec{F} \cdot d\vec{A} \approx \nabla \cdot \vec{F}(P)V \approx \iiint_C \nabla \cdot \vec{F} dxdydz}$

Example of using divergence to calculate flux: Unit hemisphere with $\vec{E} = \langle yz, xz, xy \rangle$:
Answer: $\nabla \cdot \vec{E} = \frac{\delta}{\delta x} yz + \frac{\delta}{\delta y} xz + \frac{\delta}{\delta z} xy = 0$

Example of using divergence to calculate flux: Cylinder of radius R, height h, sitting on $z = 0$ with $\vec{E} = \ln(x^2 + y^2)\langle x, y, 0 \rangle$:
Answer:

- $\frac{\delta}{\delta x} E_x = \ln(x^2 + y^2) + \frac{2x^2}{x^2+y^2}$. Similar for E_y.
- Transform to polar: $E_x + E_y = \ln(r^2) + \frac{2r^2\cos(\theta)^2 + r^2\sin(\theta)^2}{r^2} = 2\ln(r) + 2$
- Set up the integral, remembering the Jacobian: $\Phi = 2\pi \int_{z=0}^{h} \int_{r=0}^{r=R} [2\ln(r) + 2] r dr d\theta$

- Working it out, with identity $\int x\ln(x) = -\frac{x^2}{4} + \frac{x^2}{2}ln(x)$, you get $\Phi = 4\phi R^2]\ln(R)h$

Example of using divergence to calculate flux: Unit sphere at origin with $\vec{E} = (x^3+y^3)\hat{\imath} + (z^3+y^3)\hat{\jmath} + (x^3+z^3)\hat{k}$
Answer:

- $E_x + E_y + E_z = 3x^2 + 3y^2 + 3z^2 = 3\iiint \rho^2 dxdydz$
- Each $d\rho$ is a sphere of volume $4\pi f(\rho)^2 = 4\pi\rho^4$
- So the integral is $12\pi\int_{\rho=0}^{\rho=1}\rho^4 = \frac{12\pi}{5}$

Example of using divergence to calculate flux: $\vec{F} = (\cos(z)+x^2)\hat{\imath}, +(xe^{-z})\hat{\jmath} + (\sin(y)+x^2z)\hat{k}$ on parabaloid $z = x^2+y^2, z \leq 4$ with top $x^2+y^2 \leq 4, z = 4$

- $\iiint_R \nabla\cdot\vec{E} = \int_{z=r^2}^4\int_{x^2+y^2=0}^2 (y^2+x^2)dxdydz$
- $\iiint_R \nabla\cdot\vec{E} = \int_{\theta=0}^{2\pi}\int_{r=0}^2\int_{z=r^2}^4 (r^2\cos^2(\theta) + r^2\sin^2(\theta))rd\theta drdz = r^3 d\theta drdz$
- $= 2\pi\int_{r=0}^2 4r^3 - r^5 = 2\pi[r^4 - \frac{r^6}{6}]_0^2 = \frac{32}{3}\pi.$

What's crazy: Evaluating divergence of a point charge $\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{x}}{\|\vec{x}\|^3}$

- $\frac{\delta}{\delta x}E_x = \frac{Q}{4\pi\epsilon_0}\frac{\delta}{\delta x}x(x^2+y^2+z^2) = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$
- E_y, E_z follow symmetrically.
- The sum is infinite at the origin and zero everywhere else
- Therefore, they had to invent a δ function that is infinite at origin, 0 elsewhere, and $\iiint_{\mathbb{R}^3}\delta(\vec{x}) = 1$

12.4 3.8: Divergence and Fluids

Looking back to section 3.1, this hydrostatic force function should follow similar patterns to flux: $\boxed{\vec{F}_{tot} = \iint_S p\hat{n}dA}$.

Extended Example: A round ball of radius R, center at depth h , with force $\vec{F}_{tot} = p_0\iint_S[1-\frac{z}{h}]\hat{n}dA$.

- $\hat{n} = \frac{\langle x,y,z\rangle}{R}$
- For the integral, note that x, y are completely symmetric around z axis, so they contribute 0.
- For the integral, we're then looking at $\frac{p_0}{R}\iint_S[1-\frac{z}{h}]zdA$

- Use spherical coordinates: $\frac{p_0}{R}\iint_S [1-\frac{R\cos(\phi)}{h}]R\cos(\phi)dA$
- Working through $dA = \sqrt{1+f_x^2+f_y^2}dxdy$ with $f = R-\sqrt{R^2+x^2+y^2}$, we get $dA = \frac{dxdy}{\sqrt{(1-\frac{x^2+y^2}{R^2})}}$
- This dA term, in spherical coordinates, becomes $R^2\sin(\phi)d\phi d\theta$
- Combining and substituting $u = \cos(\phi)$, this integral is $-\frac{4\pi R^3}{3}\frac{p_0}{h}\hat{k}$

The neat idea: $F_{tot} = \frac{4\pi R^3}{3} \times -\frac{p_0}{h}\hat{k}$ is really "ball's volume" times a constant.

- $\frac{4\pi R^3}{3} \times -\frac{p_0}{h}\hat{k}$
- $= \iiint_B 1dxdydz \times -\frac{p_0}{h}\hat{k}$
- $= \iiint_B (-\frac{p_0}{h})\hat{k}dxdydz$, with $p = p_0[1-\frac{z}{h}]$
- $= \iiint_B (\frac{\Delta p}{\delta z})\hat{k}dxdydz$
- $= \iiint_B \nabla \cdot pdxdydz$
- So the upshot is the divergence theorem again: $\boxed{\iint_S p\hat{n}dA = \iiint_B \nabla\cdot pdxdydz = \iiint_B \nabla\cdot pd\vec{x}}$

Final example three ways: "oxygen flow" (really, flux) through ball of radius R at origin, under field $J = j_0\hat{i}$.

- Intuitive: what comes in at (-x, y, z) goes out at (x, y, z), so total is zero.
- Flux integra under spherical: $\iint \vec{F}\hat{n}dA = \iint_S j_0\hat{i}\cdot\frac{\langle x,y,z\rangle}{R}dA = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} j_0\cos(\theta)\sin(\phi)R^2\sin(\theta)d\theta d\phi = 0 = \frac{\pi j_0 R^2}{2}\int_{\theta=0}^{2\pi}\cos(\theta)d\theta = 0$
- Divergence: $\nabla\cdot\vec{J} = \frac{\delta}{\delta x}j_0 + 0 + 0 = 0, so \iiint_B 0 = 0.$

12.5 3.9: Flows and Divergence

Main idea: Divergence ($\nabla \cdot \vec{V}$) measures how much the flow changes volumes at that point.

Example: What is the function described by field of velocity vectors $\vec{V}(\vec{x}) = \langle -y, x\rangle$?

- $x'(t) = -y, y'(t) = x$
- $\Rightarrow x''(t) = -y' = -x, y''(t) = x' = -y$
- $\Rightarrow x = A\cos(t)+B\sin(t), y = C\cos(t)+D\sin(t)$, work it out to $x = \cos(t), y = \sin(t)$

Idea: dump a $dA = s_1$ by $s_2 = \Delta x\hat{\imath} \times \Delta y\hat{\jmath}$rectangle into the flow and see how it deforms over time. Over a long time, it'll distort a lot, but consider for Δt:

- dA has sides of length $\Delta x, \Delta y$ but area of dA: $\|s_1 \times s_2\|$ (cross product norm is parallelogram area)
- What is side s_1 after Δt? The starting point plus (how the endpoint moves minus how the start point moves): $\vec{s_1} + \Delta t[\vec{V}(x_0 + \Delta x, y_0) - \vec{V}(x_0, y_0)]$
- Expanding the iterated s_1, which we call s_1' out: $\vec{s_1}' = \Delta x[(1 + \Delta t\frac{\delta V_x}{\delta x})\hat{\imath} + \Delta t\frac{\delta V_y}{\delta x}\hat{\jmath}]$. Do the same for s_2' and work out in 3D: $s_1' \times s_2' \approx \Delta x\Delta y[\hat{k} + \Delta t[(V_x)_x + (V_y)_y]\hat{k}$
- We end up with $s_1' \times s_2' \approx \Delta x\Delta y[1 + \Delta t\nabla \cdot \vec{V}]\hat{k}$, so vs. original area $\Delta x\Delta y$, the ratio is $1 + \Delta t\nabla \cdot \vec{V}$
- This means **divergence $\nabla \cdot \vec{V}$ is proportional to the change in area due to the flow**!

An **incompressible** field preserves volume under flow (so $\nabla\cdot\vec{V} = 0$), like $\langle y, z, x\rangle, \langle 0, 2\sqrt{x}, 0\rangle, \langle x, y, -2z\rangle$.

A cool interactive on the page shows how a sphere migrating its points via $\langle x, y, z\rangle$ grows and changes volume, while one under $0.3\langle y, z, x\rangle$ distorts but doesn't.

13 Chapter 4: Work, Line Integrals, Stokes's Theorem

13.1 4.1: Work Part I

Energy: U is a position-consuming function, an example of a potential. Energy U for charge amount q in field $\vec{E}$ is $\boxed{-q\vec{E} = \nabla U}$.
Example: Right-pointing constant field $\vec{E} = E_0\hat{\imath}$ means $(\frac{\delta}{\delta x}\hat{\imath} + \frac{\delta}{\delta y}\hat{\jmath} + \frac{\delta}{\delta z}\hat{k})U = -qE_0\hat{\imath} \Rightarrow U = -qE_0x$

$\boxed{\textbf{Work} \text{ is change in energy, e.g. } W_{field} = -[U(\vec{x_f}) - U(\vec{x_0})].}$ If the charge flows with the field, then the field is doing positive work. If the charge flows against the field, the field is doing negative work. So in the above case, $W_{field} = -[U(x_f, 0, 0)) - U(x_0, 0, 0)] \Rightarrow -[-qE_0x_f - -qE_0x_0 = qE_0(x_f - x_0)$.

Continuing the example, moving from $(x_0, 0, z_0)$ to $(x_f, 0, z_f)$ in $\vec{E} = E_0\hat{\imath}$: only the x-coordinate affects the energy, so $U(x) = -qE_0x$ and $W_{field} = qE_0(x_f - x_0)$. If s is the distance between the two, then $\cos(\theta) = \frac{|x_f - x_0|}{s} \Rightarrow W_{field} = qE_0s\cos(\theta)$.

Expanding the example, consider $\vec{E}(\vec{x}) = \vec{E_0}$, a constant that may not be aligned just with x-axis. Then $-q\vec{E} = \nabla U \Rightarrow \nabla(-q\vec{E_0}\vec{x} + C) = -q\vec{E} \Rightarrow U = -q\vec{E_0}\vec{x} + C$. Can also solve the diff eq, more generally. This also means that, still, $W_{field} = q\vec{E_0}(\vec{x_f} - \vec{x_0})$.

Big idea: Though most fields aren't constant, they are near-constant between a small displacement Δx.

$$W_{field} = q\vec{E} \cdot \Delta\vec{x} \tag{85}$$

$$= \sum q\vec{E}(\vec{x}(t_i)) \cdot [\vec{x}(t_{i+1}) - \vec{x}(t_i)] \tag{86}$$

$$= \sum q\vec{E}(\vec{x}(t_i))\frac{\Delta x(t_{i+1})}{\Delta t}\Delta t = \int_a^b q\vec{E}(\vec{x}(t)) \cdot \frac{d\vec{x}}{dt}dt \tag{87}$$

So we can take a **line integral** to measure work over a path in a field.

13.2 4.2: Work Part 2

In general, the **work** for moving a charge q along path $\vec{x}(t)$ through field $\vec{E}(\vec{x})$ is $\boxed{W = \int_a^b q\vec{E}(\vec{x}(t)) \cdot \frac{d\vec{x}}{dt}dt}$.

More generally, $q\vec{E}$ is just a force, so we're looking at $\boxed{W_{field} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \frac{d\vec{x}}{dt}dt}$

Example: $\vec{x}(t) = \langle t+1, t+1, t\rangle, t \in [0, 8], \vec{E}(x, y, z) = \frac{\lambda}{2\pi\epsilon_0}\frac{x\hat{i}+y\hat{j}}{x^2+y^2}$

$$W = \frac{q\lambda}{2\pi\epsilon_0}\int_{t=0}^{8}\frac{\langle t+1, t+1, t\rangle \cdot \langle 1, 1, 0\rangle}{2(t+1)^2} \tag{88}$$

$$= \frac{q\lambda}{2\pi\epsilon_0}\int_{t=0}^{8}\frac{1}{t+1} \tag{89}$$

$$= \frac{q\lambda}{2\pi\epsilon_0}\int_{u=1}^{9}\frac{1}{u} \tag{90}$$

$$= \frac{q\lambda}{2\pi\epsilon_0}\ln(9) \tag{91}$$

However, what happens if we keep the endpoints but change the path?

Extended example: $\vec{x}(t) = \langle\sqrt{2}t\cos(\frac{\pi t}{4}), \sqrt{2}t\sin(\frac{\pi t}{4}), t-1\rangle$, still with $\vec{E}(x, y, z) = \frac{\lambda}{2\pi\epsilon_0}\frac{x\hat{i}+y\hat{j}}{x^2+y^2}$.

$$\vec{E}(\vec{x}(t)) \cdot \frac{d\vec{x}}{dt} = ... = 2t \tag{92}$$

$$x^2 + y^2 = 2t^2 \tag{93}$$

$$W = \frac{q\lambda}{2\pi\epsilon_0}\int_{1=1}^{9}\frac{1}{t} \tag{94}$$

$$= \frac{q\lambda}{2\pi\epsilon_0}\ln(9) \tag{95}$$

So it looks like there *might be path independence here.* Consider: rubbing your hands together - is there more work done oscillating and ending at the initial position than doing nothing? (Clearly yes).

Example: Work due to friction. $\vec{F} = -\gamma\frac{d\vec{x}}{dt}, \gamma > 0$. Move from $(0,0,0) \to (1,0,0)$ via $\vec{x}(t) = \langle t, 0, 0\rangle$. If $\gamma = 1$, what is $\int_a^b \vec{F}(\vec{x}(t)) \cdot \frac{d\vec{x}}{dt}dt$?

Answer: $-\int_{t=0}^{1} \frac{d\vec{x}}{dt}^2 dt = -\int_{t=0}^{1}\langle 1,0,0\rangle^2 = -1$ We lose 1 unit of energy.

To illustrate that there's not path independence always, consider an oscillating object following $\vec{x}(t) = \langle t, \sin(n\pi t), \sin(n\pi t)\rangle, t \in [0,1]$. Take $\gamma = 1$ so $\vec{F} = \vec{x}'(t)$.

Answer: $-\int_{t=0}^{1} \vec{x}'(t) \cdot \vec{x}'(t)dt = -\int_{t=0}^{1}(1 + 2n^2\pi^2\cos^2(n\pi t))dt = -\int_{t=0}^{1}(1 + 2n^2\pi^2\frac{1}{2}(1 + \cos(2n\pi t))dt = -\int_{t=0}^{1}(1 + n^2\pi^2 + 2n^2\pi^2\cos(2n\pi t))dt = -[t + tn^2\pi^2 = +n\pi\sin(2n\pi t)]_0^1 = -[1 + n^2\pi^2]$

Example: Spring force $\vec{F} = -\frac{\|x-l_0\|}{\|x\|}\vec{x}$ along path $\vec{x}(t) = \langle 0, 1-t, 2t\rangle, t \in [0,1]$

Answer: $\int_{t=0}^{1} \vec{F}(\vec{x}(t))\frac{d\vec{x}}{dt}dt = -\int_{t=0}^{1} \frac{\sqrt{5t^2-2t+1}-1}{\sqrt{5t^2-2t+1}}\langle 0, 1-t, 2t\rangle \cdot \langle 0, -1, 2\rangle = \int_0^1 [1 - (5t^2 - 2t + 1)^{\frac{-1}{2}}](-1+5t)dt = -[-t + \frac{5}{2}t^2 - (5t^2 - 2t + 1)^{\frac{1}{2}}]_0^1 = -\frac{1}{2}$

Example: *SAME* Spring force $\vec{F} = -\frac{\|x-l_0\|}{\|x\|}\vec{x}$ and *SAME* endpoints but along path $\vec{x}(t) = \langle 0, \cos(t), 2\sin(t)\rangle, t \in [0,1]$

Answer: $\int_{t=0}^{\pi/2} \vec{F}(\vec{x}(t))\frac{d\vec{x}}{dt}dt = -\int_{t=0}^{\pi/2}(1 - \frac{1}{\|x\|})\langle 0, \cos(t), 2\sin(t) \cdot \langle 0, -\sin(t), 2\cos(t)\rangle = -3\int_0^{\pi/2}\sin(t)\cos(t)(1 - \frac{1}{\sqrt{1+3t^2}}dt = 3\int_{t=0}^{\pi/2} -\sin(t)\cos(t) + 3\int_{t=0}^{\pi/2}\sin(t)\cos(t)\frac{1}{\sqrt{1+3\sin^2(t)}} = 3[\frac{1}{2}\cos^2(t)]_0^{\pi/2} + 3[\frac{2}{6}(1 + 3\sin^2(t))^{1/2}]_0^{\pi/2} = -1/2$

So sometimes path does not matter (electricity and spring) but sometimes it seems to matter (friction). Something about the "swirl" of the field (curl)? Addressed upcoming.

13.3 4.3: Line Integrals

A line integral is

- Most generally: $\int_{s=0}^{s=l} f(\vec{x}(s))ds$ or $\int_C f ds$.
- Built out of function heights h_i over curve snippet lengths Δs_i : $\sum_i h_i \Delta s_i$.
- More practically written as $\sum f(x_i, y_i)\sqrt{[\Delta x]^2 + [\Delta y]^2}$
- Riemanned up: $\int_{s=0}^{s=l} f(\vec{x}(s))ds$
- Practical integral based off of some t : $\int_C f(\vec{x}(t))\|\vec{x}'(t)\|dt$

Example: Area of curtain, height $= f(x,y) = y^2$, base is origin circle of radius 2 on xy-plane.

- $A = \int_{t=0}^{t=2\pi} f(x,y) \| \frac{d<2\cos(t),2\sin(t),0>}{dt} \| dt$
- $A = \int_{t=0}^{t=2\pi} 4\sin^2(t) * 2 = 4\int_{t=0}^{t=2\pi} (1-\cos(2t)) = 4[t - \frac{\sin(2t)}{2}]_0^{2\pi} = 8\pi$

Example: Geometric area of $f(x,y) = 4y^3$ over the curve $x = \frac{y^3}{3}, (\frac{-1}{3}, 1) \to (\frac{1}{3}, 1)$

- $\vec{x}(t) = \langle t^3/3, t\rangle \Rightarrow \vec{x}'(t) = \langle t^2, 1\rangle \Rightarrow \|\vec{x}'(t)\| = \|\sqrt{1+t^4}\|$
- Note that this is an odd function: the part where $y \in [-1, 0]$ is a negative of [0,1]. So we'll double the right half.
- $A = 2\int_{t=0}^{t=1} 4t^3\sqrt{1+t^4}dt = [\frac{4}{3}(1+t^4)^{\frac{3}{2}}]_0^1 = \frac{4}{3}[2\sqrt{2}-1]$

Example: Moment of interatia around z-axis: $\int [x^2+y^2]\rho ds$ for $\vec{x}(t) = \langle 2\sin(t), 2\cos(t), 3t\rangle t \in [0, 2\pi]$ if $\rho = \frac{1}{2\pi\sqrt{13}}$:

- $\vec{x}'(t) = \langle 2\cos(t), 2\sin(t), 3 \Rightarrow \|\vec{x}'(t)\| = \sqrt{4+9}$
- $x^2 + y^2 = (2\sin(t))^2 + (2\cos(t))^2 = 4$
- $A = \rho * \int_{t=0}^{t=2\pi} 4\sqrt{13} = \frac{1}{2\pi\sqrt{13}} * 2\pi * 4\sqrt{13} = 4$

Example: Infinite wire of current going up on the z-axis. Field equation, for (generated?) field $\vec{B}$, penetrated region bounded by curve C, unit tangent (to the curve) $\hat{T}$, constant μ_0 and current amount I: $\boxed{\int_C [\vec{B} \cdot \hat{T}]ds = \mu_0 I}$. What is $\|\vec{B}\|$ at some distance from the z-axis r?

- C should be a circle of radius r. Then, $ds = \frac{d\vec{x}}{dt}dt = r\|\cos(t), \sin(t)\| = r$
- $\vec{B} = \|B\|\hat{T}$ by definition I suppose.
- $\int_{t=0}^{t=2\pi} [\|B\|\hat{T} \cdot \hat{T}]r dt$
- $\|B\| * 2\pi r = \mu_0 I$
- $\Rightarrow \|\vec{B}\| = \frac{\mu_0 I}{2\pi r}$

13.4 4.4: Path independence

When do we care only about the endpoints and not the path? Examples include the spring and the electric charge, and counterexamples include motion under friction.

Note that integrals of the form $\int_C \frac{df}{ds}ds = \int_{s=0}^{s=l} f'(s)ds = f(x(s=l)) - f(x(s=0))$ are path independent (fundamental theorem of calculus). **So if you can find a legit**

antiderivative of our function, it is path independent. Or, if $\vec{F}$ is a **gradient vector field** ($\vec{F} = \nabla f$), then the line integral depends only on the endpoints of C.

Note: If $f(s) = f(\vec{x}(s))$, then $\int_C f'(s) = \int_C f'(\vec{x}(s)) = \int_{s=0}^{s=l} \frac{\delta f}{\delta x}(x(s))x'_x(s) + \frac{\delta f}{\delta y}(x(s))x'_y(s) + \frac{\delta f}{\delta z}(x(s))x'_z(s)$ or $\boxed{\int_{s=0}^{s=l} \nabla f(x(s)) \cdot x'(s)ds = \int_C f'(s)}$. *This looks less like a magical theorem and more of a description of a componentwise derivative, line-integrated over ds like usual.*

Example: if $\vec{E} = \frac{\lambda}{2\pi\epsilon_0} \frac{x\hat{i}+y\hat{j}}{x^2+y^2}$, find f such that $\nabla f = \vec{E}$.

- Note: This is just the anti- and derivative cycle technique.
- What's on the $\hat{i}$? That's $\frac{\delta f}{\delta x} = \frac{\lambda}{2\pi\epsilon_0} \frac{x}{x^2+y^2}$
- What's on the $\hat{j}$? That's $\frac{\delta f}{\delta y} = \frac{\lambda}{2\pi\epsilon_0} \frac{y}{x^2+y^2}$
- Anti-derivative on the x: $f = \frac{\lambda}{4\pi\epsilon_0} \ln(x^2 + y^2) + h(y)$
- Re-differentiate for y: $\frac{\delta f}{\delta y} = \frac{\lambda}{2\pi\epsilon_0} \frac{y}{x^2+y^2} + h'(y)$
- So $h(y)$ is constant, and $f = \frac{\lambda}{4\pi\epsilon_0} \ln(x^2 + y^2) + C$

Example: Another cycle: if $\vec{F} = (2x - 3y)\hat{i} + (6y - ax)\hat{j}$, what's an a to allow for $\nabla f = \vec{F}$?

- Note: This is just the anti- and derivative cycle technique again
- What's on the $\hat{i}$? That's $\frac{\delta f}{\delta x} = 2x - 3y$
- What's on the $\hat{j}$? That's $\frac{\delta f}{\delta y} = 6y - ax$
- Anti-derivative on the x: $f = x^2 - exy + h(y)$
- Re-differentiate for y: $\frac{\delta f}{\delta y} = -3x + h'(y) = -3x + 6y$. $a = -3$.

Example: Spring: if $\vec{F} = -(\|\vec{x}\| - l_0)\frac{\vec{x}}{\|\vec{x}\|}$, find f such that $\nabla f = \vec{F}$.

- Hint given: $\nabla(\frac{1}{2}\|\vec{x}\|^2) = \vec{x} \Rightarrow \nabla(\|\vec{x}\|) = \frac{\vec{x}}{\|\vec{x}\|}$
- Rewrite as $-\vec{x} + l_0\frac{\vec{x}}{\|\vec{x}\|}$
- KEEP the nablas!
- Use hints on each term: $\vec{F} = -\nabla(\frac{1}{2}\|\vec{x}\|^2) + l_0\nabla(\|x\|)$
- $\vec{F} = \nabla(-\frac{1}{2}\|\vec{x}\|^2 + l_0\|x\|)$

- Note: Second term worked out through chain rule, or you can just do the hand-math on $\sqrt{x^2 + y^2 ...}$

- **The magic**: Now we can use f to find the work from earlier problems: $f(0,0,2) - f(0,1,0) = -\frac{1}{2}$!

Curve orientation: pick two opposite parametrizations of the curve C_+ and C_- (say, "going right" and "going left"). We can use right-hand rule, where, if curve in some xy plane, right thumb points along $\hat{k}$ and C_+ curls with fingers.

We know that $\int_{C_+} \vec{F} \cdot d\vec{x} = -\int_{C_-} \vec{F} \cdot d\vec{x}$ and therefore, $\int_{C_+ + C_-} \vec{F} = 0$

We can say $\boxed{\text{If } \int_C \vec{F} \cdot d\vec{x} \text{ depends only on endpoints of } C \text{ then } \vec{F} \text{ is a gradient field.}}$

- Basic idea: Similar to the fundamental theorem $g(x) = \frac{d}{dx} \int_{t=a}^{t=x} g(t)dt$, we can prove $\vec{F}(\vec{u}) = \nabla[\int_{\vec{a}}^{\vec{u}} \vec{F} \cdot d\vec{x}]$.

- Plan: prove directional derivative of $\int \vec{F} \cdot d\vec{x}$ along $\vec{v}$ is $\vec{v} \cdot \vec{F}(\vec{u})$ for any $\vec{v}$, which proves the answer is $\vec{F}(\vec{u})$

- Neat idea: We're looking for $\int_{\vec{a}}^{\vec{u}+h\vec{v}} \vec{F} \cdot d\vec{x} - \int_{\vec{a}}^{\vec{u}} \vec{F} \cdot d\vec{x} = \int_{\vec{a}}^{\vec{u}+h\vec{v}} \vec{F} \cdot d\vec{x} + \int_{\vec{u}}^{\vec{a}} \vec{F} \cdot d\vec{x} = \int_{\vec{u}}^{\vec{u}+h\vec{v}} \vec{F} \cdot d\vec{x}$!

- Because of path independence, we can parametrize this as $\int_{t=0}^{t=h} \vec{F}(\vec{u}+t\vec{v}) \cdot \frac{d}{dt}[\vec{u}+t\vec{v}]dt$

- $= \vec{v} \cdot \int_{t=0}^{t=h} \vec{F}(\vec{u}+t\vec{v})dt$, which under $\lim \frac{1}{h}$ and L'Hopital's, becomes $\vec{v} \cdot \vec{F}(\vec{u})$.

Also:

$\boxed{\int_C \vec{F} \cdot d\vec{x} \text{ depends only on the endpoints of } C \text{ if and only if } \int_L \vec{F} \cdot d\vec{x} = 0 \text{ for any closed loop } L}$

- If F depends only on the endpoints, then any closed loop L is basically a C_+ and a C_-, so the circuit integrates to 0.

- If every loop is 0, then any two curves with the same endpoints form a loop (after reversing one), so paths are independent.

Note that the two theorems together imply $\boxed{\vec{F} = \nabla f \Leftrightarrow \int_L \vec{F} \cdot d\vec{x} = 0 \text{ for a connected domain } D.}$ Such an F is called **conservative**.

13.5 3.5: Curl

Conservative fields, where any loop line-integrates to zero, get their name because no work is performed over such a loop.

Main idea: It's hard to show every loop is zero. Instead, lets look at very small loops and go from there. Proof idea:

- Pick a point $\vec{p}$ in a connected (no holes) domain D.
- Use the right-hand rule (for example) and build a small circular loop L in a plane around D.
- Consider the linear approximation: The approximate field around p is the exact field at p plus the derivative at p multiplied by the delta from p.
- So, this means $\vec{F}(\vec{x}(t)) \approx \vec{F}(\vec{p}) + D\vec{F}(\vec{p})(\vec{x}(t) - \vec{p})$
- Substitute into line integral: $\int_L [\vec{F}(\vec{p}) + D\vec{F}(\vec{p})(\vec{x}(t) - \vec{p})]dx = 0$
- The first term is zero because it's a conservative field: $\vec{F}(\vec{p}) = \nabla(\vec{F}(\vec{p})\vec{x})$
- The integral of the second term relates to $\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}$:
  - **Goal**: Calculate $\frac{1}{\pi\epsilon^2}\int_L D\vec{F}(p)(\vec{x} - \vec{p}) \cdot d\vec{x}$
  - Consider a loop in the xy plane only (meaning $\hat{n} = \hat{i}$). Then $\vec{x}(t) = \langle \epsilon\cos(t), \epsilon\sin(t), 0\rangle$
  - $\vec{x}'(t) = \langle -\epsilon\sin(t), \epsilon\cos(t), 0\rangle$
  - $D\vec{F}(\vec{p})(\vec{x}(t) - \vec{p}) \cdot d\vec{x} = \frac{\delta F_x}{\delta x}(\vec{p})(\vec{x}_x - \vec{p}_x)\frac{dx}{dt} + \frac{\delta F_y}{\delta x}(\vec{p})(\vec{x}_x - \vec{p}_x)\frac{dx}{dt} + \frac{\delta F_x}{\delta y}(\vec{p})(\vec{x}_y - \vec{p}_y)\frac{dy}{dt} + \frac{\delta F_y}{\delta y}(\vec{p})(\vec{x}_y - \vec{p}_x)\frac{dy}{dt}$
  - (Note that the $-p_x, -p_y$ terms will integrate to zero, and that $F_z = 0$).
  - **A-ha**: Also note that same-variable terms disappear too: $\int_L \frac{\delta F_x}{\delta x}(\vec{p})(\vec{x}_x * \frac{dx_x}{dt}) = \int_{t=0}^{t=2\pi} \epsilon\cos(t) * -\epsilon\sin(t) = -\epsilon^2 \int_{t=0}^{t=2\pi} \sin(t) = 0$
  - So we're left with $\int_{t=0}^{t=2\pi} \frac{\delta F_x}{\delta y}\epsilon\sin(t)(-\epsilon(\cos(t))) + \frac{\delta F_y}{\delta x}\epsilon\cos(t)\epsilon(\cos(t))$
  - Using $\int_{t=0}^{t=2\pi} \cos^2(t) = \int_{t=0}^{t=2\pi} \frac{1}{2} - \frac{\cos(2t)}{2} = \pi$ (and given the curve shape, same integral result for sin)...
  - $\epsilon^2(\frac{\delta F_y}{\delta x}(p)\int_{t=0}^{t=2\pi} \cos^2(t)dt - \frac{\delta F_x}{\delta y}(p)\int_{t=0}^{t=2\pi} \sin^2(t)dt)$
  - $= \pi\epsilon^2[\frac{\delta F_y}{\delta x}(p) - \frac{\delta F_x}{\delta y}(p)]$
  - **Don't Forget**: What we proved is that the line integral around that point is (a piece of) the curl times the area of the contained shape!
- Do this again for xz and yz planes ($\hat{n} = \hat{j}, \hat{n} = \hat{k}$), and we end up with $\frac{1}{\pi\epsilon^2}\int_L D\vec{F}(p)(\vec{x} - \vec{p}) =$ (many equivalent forms)
  - $\{-\frac{\delta F_y}{\delta z} + \frac{\delta F_z}{\delta y}, \hat{n} = \hat{i}; -\frac{\delta F_z}{\delta x} + \frac{\delta F_x}{\delta z}, \hat{n} = \hat{j}; -\frac{\delta F_x}{\delta y} + \frac{\delta F_y}{\delta x}, \hat{n} = \hat{k}\}$

$$- \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{pmatrix} \cdot \hat{n}$$

$$- (\nabla \times \vec{F}) \cdot \hat{n}$$

- So the **curl** is defined as $(\nabla \times \vec{F}) \cdot \hat{n}$, sort of like the **divergence** is $\nabla \cdot \vec{F}$
- *TODO What's the deal with the $\hat{n}$ part?*

A **potential** is a function f that has a corresponding conservative vector field $\vec{F} = \nabla f$. (Note: Wouldn't any f fit the bill here?).

The big theorem for conservative fields $\vec{F}$: $\boxed{\nabla \times \vec{F} = \vec{0} \Leftrightarrow \int_L \vec{F} d\vec{x} = 0 \Leftrightarrow \vec{F} = \nabla f}$

Example: A disk rotating around z axis with rotational velocity ω. What's the curl?

- $\vec{r} =$ some $< x, y, z >$.
- Velocity vector on disk $\vec{v}$ is perpendicular to disk position r (perp to both xy position and z), and is $[\omega\hat{k}] \times \vec{r}$.
- This cross product $\vec{v} = \omega x \hat{j} - \omega y \hat{i}$.
- So curl is $\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \omega y & \omega x & 0 \end{pmatrix} = 2\omega\hat{k}$
- **Curl Intuition**: So the curl "curls" around the axis of rotation, gets bigger with a faster rotation, and would reverse if the direction reversed.

Example: A river flowing with field $\vec{F} = e^{-x^2}\hat{i}$ What's the curl?

- $\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ e^{-x^2} & 0 & 0 \end{pmatrix} = 0$
- **Intuition**: If you put a cross-shaped boat in the water, axis aligned, it wouldn't twist.

Example: A river flowing with field $\vec{F} = e^{-x^2}\hat{j}$ What's the curl?

- $\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 0 & e^{-x^2} & 0 \end{pmatrix} = -2xe^{-x^2}\hat{k}$
- **Intuition**: If you put a cross-shaped boat in the water, axis aligned, it WOULD twist, since the upward flows have different strengths on different sides of the boat.

13.6 4.6: Stokes's Theorem

Like the Divergence Theorem stacking cubes, Stokes's Theorem can be built up from small pieces too (rectangular loops). Proof:

- Note: "δ" means the boundary of a region.
- Same-oriented square loops R_1, R_2 can be merged to share two canceling edges (going in opposite directions) and thus one bigger loop.
- Therefore, $\int_{\delta[R1+R_2]} \vec{F} \cdot d\vec{x} = \int_{\delta R_1} \vec{F} \cdot d\vec{x} + \int_{\delta R_2} \vec{F} \cdot d\vec{x}$
- Before, in section 4.5, we proved that $\frac{1}{A_i} \int_{\delta r_i} \vec{F} \cdot d\vec{x} = (\nabla \times \vec{F}(p_i)) \Rightarrow \int_{\delta r_i} \vec{F} \cdot d\vec{x} = A_i(\nabla \times \vec{F}(p_i))$
- so $\int_{\delta[r_1+r_2\ldots]} \vec{F} \cdot d\vec{x} = \sum A_i \nabla \times \vec{F}(p_i)$
- As we increase the granularity and shrink A_i to zero, this becomes **Stokes's Theorem**: $\boxed{\int_{\delta S} \vec{F} \cdot d\vec{x} = \iint_S \nabla \times \vec{F} \cdot d\vec{A}}$. So the line integral of the skirt is the same as the curl integrated over the surface.
- So this means that closed surfaces sum to zero curl, and a conservative fields sums to zero curl $(\nabla \times (\nabla F) = \nabla \times \vec{0} = \vec{0}$.
- TODO: What happened to $\hat{n}$?

Example: The hemisphere $z = R^2 - x^2 - y^2$ with equator intersecting the xy-plane at circle C, with "arrows" oriented counterclockwise viewed from the top of $\hat{k}$. If $\vec{F} = -\frac{1}{3}y^3\hat{\imath} + \frac{1}{3}x^3\hat{\jmath} + z\hat{k}$, compute line integral $int_C \vec{F} \cdot d\vec{x}$.

Line integral technique:

- $x(t) = \langle R\cos(t), R\sin(t), 0\rangle$, so $\vec{F} \cdot d\vec{x} = \frac{1}{3}R^4\sin^4(t) + \frac{1}{3}R^4\cos^4(t)$
- Given identity $\int_{\theta=0}^{2\pi} \cos^4(\theta) = \int_{\theta=0}^{2\pi} \sin^4(\theta) = \frac{3\pi}{4}$
- Then integral quickly becomes $\frac{1}{3}R^4 * (\frac{3\pi}{4} * 2) = \frac{R^4\pi}{2}$

Curl technique:

- $\det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ -\frac{1}{3}y^3 & \frac{1}{3}x^3 & 0 \end{pmatrix} = (x^2 + y^2)\hat{k}$
- $f = z + x^2 + y^2 = R^2$ (level set). Gradient $\nabla f = \langle 2x, 2y, 1\rangle$ is normal by level set and $\hat{n} = \frac{\langle 2x,2y,1\rangle}{\sqrt{4x^2+4y^2+1}}$, so $\hat{k} \cdot \hat{n} = \frac{1}{\sqrt{4x^2+4y^2+1}}$

- $dA = \|\vec{x}_x \times \vec{x}_y\| = \|\langle 1,0,-2x\rangle \cdot \langle 0,1,-2y\rangle\| = \sqrt{4x^2+4y^2+1}$
- $\iint_S \nabla \times \vec{F} \cdot d\vec{A} = \iint_S (x^2+y^2)\hat{k} \cdot \hat{n} dA = \iint_S (x^2+y^2)(\hat{k} \cdot \hat{n}) dA$
- $= \iint_S [x^2+y^2] \frac{1}{\sqrt{4x^2+4y^2+1}} \sqrt{4x^2+4y^2+1}$
- Switch to polar, don't forget Jacobian for dA: $= \int_{r=0}^{R} \int_{\theta=0}^{2\pi} (r^2\cos^2(\theta) + r^2\sin^2(\theta)) r dr d\theta$
- $= 2\pi \int_{r=0}^{R} r^3 = \frac{2\pi R^4}{4} = \frac{R^4\pi}{2}$

Another feature of Stokes: If finding the surface area of a figure S only depends on boundary δS, $\boxed{\text{you can switch to another figure } S' \text{ as long as the boundary doesn't change!}}$

Example: Same unit hemisphere, but raised by a unit cylinder, with $\vec{F} = \langle e^{yz}, x\cos(z^3) + xze^{yz}, cos(sin(xyz))\rangle$

- The a-ha is that a unit disk will have the same boundary, so set $z = 0$
- This reduces to $\vec{F} = 1, x, 1$, and easy to compute $\nabla \times \vec{F} = 1 * \hat{k}$
- $\hat{n} = \hat{k}, so \iint_S \hat{n} \cdot \hat{k} = \iint_S 1 = \pi$ by area of a circle!

Another implication: a closed surface (say sphere S) has $\iint_S \nabla \times \vec{F} \cdot d\vec{A} = 0$ by the argument:

- Separate into two hemispheres S_+, S_- with skirts running in opposite directions.
- These have $\iint_{S_+} \nabla \times \vec{F} \cdot d\vec{A} = \int_{C_+} \vec{F} \cdot d\vec{x} = -\int_{C_-} \vec{F} \cdot d\vec{x} = \iint_{S_-} \nabla \times \vec{F} \cdot d\vec{A}$
- $\int_{C_+}$ and $\int_{C_-}$ therefore sum to zero.
- And by Stokes theorem, their surface integrals $\iint_{S_+}$ and $\iint_{S_-}$ do too.

Stokes's Theorem also works in 2D. Consider any such region S a 2D xy-bound object with normal $\hat{k}$.

- Write the field as $\vec{F} = \langle P(x,y), Q(x,y), 0\rangle$
- $\int_C \vec{F} \cdot d\vec{x} = \int_C Pdx + Qdy$ by definition.
- Take the curl, which is $\nabla \times \vec{F} = \frac{\delta Q}{\delta x} - \frac{\Delta p}{\delta y}$
- By Stokes's Theorem, $\int_C Pdx + Qdy = \iint_S [\frac{\delta Q}{\delta x} - \frac{\Delta p}{\delta y}] dxdy$
- **Green's theorem** $\boxed{\iint_D [\frac{\delta G_x}{\delta x} - \frac{\delta G_y}{\delta y}] dxdy = \int_C \vec{G} \cdot \hat{n} ds}$ is just a 2D divergence theorem.

  - Why? Substitute $P = -G_y, Q = G_x$ into above, see $\int_C Pdx + Qdy = \vec{G} \cdot \langle dy, -dx \rangle$. The last vector is normal to $\langle dx, dy \rangle$, and unit, so we've got it.

Gotcha Even though conservative flows have curl $= \vec{0}$, $\vec{0}$ flows CAN be non-conservative on non-simple domains. Example:

- $\vec{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \rangle$.
- Domain $D = \mathbb{R}^2 - (0, 0)$
- Curl $[\frac{\delta}{\delta x}(\frac{x}{x^2+y^2}) + \frac{\delta}{\delta y}(\frac{x}{x^2+y^2})]\hat{k} = \vec{0}$ after calcuation.
- But $\int_C \vec{F} \cdot d\vec{x}$, when changed to polar with $\vec{x} = \langle \cos(t), \sin(t), 0 \rangle$, becomes $\int_C \frac{\sin^2(t)}{1} + \frac{\cos^2(t)}{1^2} = \int_C 1 = 2\pi$
- This closed loop is not 0, so the field is not conservative, but the curl still is $\vec{0}$.
- The **gotcha** is that the domain is not simply connected like a sphere, but more like a donut with a hole in the middle.
- If we can't shrink our circle to a point, the loop doesn't have to integrate to zero. (TODO: Like in complex analysis?)

13.7 4.7: Swirls and Curls

To review:

- Divergence: How many lines enter or leave the neighborhood of the point, indicates how much volume changes over the flow.
- Divergence theorem relates Flux through a closed bounding surface and the divergence over the internal volume. Main idea: stuff inside cancels out, and what's going in or out of the skin ends up being the sum of the volume's field inside.
- Divergence theorem relates any $n-1$ and n dimension hyperspaces: $\boxed{\iint_S \vec{F} \cdot \hat{n} dA = \iiint_R \nabla \cdot \vec{F} dV}$
- Curl: How much is the flow swirling around the point (not entering or leaving). Rotational torque ends up motivating this in this chapter.
- Curl relates a line integral and the 2D-surface (possibly in higher space) it skirts.
- Stokes's Theorem: $\boxed{\int_{\delta S} \vec{F} \cdot d\vec{x} = \iint_S \nabla \times \vec{F} \cdot d\vec{A}}$
- $\square$
- **NOTE: These cross over** in the case where field is the curl of some other field: $\vec{G} = \nabla \times \vec{F}$:

- $\iint_S \nabla \times \vec{F} \cdot d\vec{A} = \iint_S \vec{G} \cdot \hat{n} dA = \iiint_R \nabla \cdot (\nabla \times \vec{F}) dV = 0$
- This is because the divergence of a curl is always zero. Can work out in a straightforward way. Not sure of intuition yet.

Example: Torque $\vec{\tau}$ on a mass at position $\vec{r}$ under angular velocity around origin $\vec{\omega}$ and and force $\vec{F} = \langle F_x, F_y, 0 \rangle$.

- Torque formula: $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{F}$ is proportional to $\frac{\delta \vec{\omega}}{dt}$
- Rod connecting mass to origin is vector $\langle l\cos(\theta_0), l\sin(\theta_0), 0 \rangle$
- Just compute the cross product: $\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ l\cos(\theta_0) & l\sin(\theta_0) & 0 \\ F_x & F_y & 0 \end{pmatrix} = l[F_y \cos(\theta_0) - F_x \sin(\theta_0)]\hat{k}$
- This indicates the torque is perpendicular to $\mathbb{R}^2$ always, confining the rotation and position there.

Example: Cross-shaped paddle wheel ($\vec{r_0}$ with spokes $\vec{r_1} ... \vec{r_4}$) in a water flow $\vec{V}$

- Torque on paddle $\vec{\tau_i} = k[\vec{r_i} - \vec{r_0}] \times [\vec{V}(\vec{r_i}) - \vec{V}(\vec{r_0})^{\perp}] = k[\vec{r_i} - \vec{r_0}] \times [\vec{V}(\vec{r_i}) - \vec{V}(\vec{r_0})]$ since the parallel part zeroes out.
- Torque on paddle $\vec{\tau_1} = k\langle l\cos(\theta), l\sin(\theta), 0\rangle \times \langle [\frac{\delta V_x}{\delta x} l\cos(\theta) + \frac{\delta V_x}{\delta y} l\sin(\theta)], [\frac{\delta V_y}{\delta x} l\cos(\theta) + \frac{\delta V_y}{\delta y} l\sin(\theta)], 0 \rangle$
- $= kl^2[\frac{\delta V_y}{\delta x}\cos^2(\theta) + [\frac{\delta V_y}{\delta y} - \frac{\delta V_x}{\delta x}]\sin(\theta)\cos(\theta) - \frac{\delta V_x}{\delta y}\sin^2(\theta)]\hat{k}$
- Recognizing that r_2, r_3, r_4 are the same as r_1 except with θ augmented by $\frac{pi}{2}, \pi, \frac{3\pi}{2}$, we eventually can sub and get $\vec{\tau} = \vec{\tau_1} + \vec{\tau_2} + \vec{\tau_3} + \vec{\tau_4} = 2kl^2[\frac{\delta V_y}{\delta x} - \frac{\delta V_x}{\delta y}]\hat{k}$
- So, $\vec{\tau}$ is proportional to $[\frac{\delta V_y}{\delta x} - \frac{\delta V_x}{\delta y}]\hat{k}$
- Fact, we define **curl in 2D** as $\boxed{curl(\vec{V}(x,y) = [\frac{\delta V_y}{\delta x} - \frac{\delta V_x}{\delta y}]\hat{k}}$
- This can just as easily be in the yz-plane as $curl(\vec{V}(y,z) = [\frac{\delta V_z}{\delta y} - \frac{\delta V_y}{\delta z}]\hat{i}$
- Note: we also define a **right-handed** system as one where vectors $\vec{v_1} \times \vec{v_2} = \vec{v_3}$, so the j one ends up being negative since $\hat{i} \times \hat{k} = -\hat{j}$

An **irrotational** field is one where the cross-shaped paddles don't spin, or where curl $= \vec{0}$.

13.8 4.8: Differential Forms

General Form of Stokes' Theorem is $\boxed{\int_M d\omega = \int_{\delta M} \omega}$. This chapter explains it.

A **1-form** takes a *vector*, outputs a *number* and is *linear*. So $\boxed{\omega : \mathbb{R}^n \to \mathbb{R}, \omega[a\vec{x} + b\vec{y}] = a\omega[\vec{x}] + b\omega[\vec{y}].}$

- 1-form: $\vec{v} \cdot \vec{x}$, since $\vec{v} \cdot (a\vec{x} + b\vec{y}) = a\vec{v} \cdot \vec{x} + b\vec{v} \cdot \vec{y}$
- Not a 1-form: $\|x\|^2$, since $\sqrt{((ax_1)^2 + (by_1))^2) + (ax_2)^2 + (by_2))^2) + ...}^2 = a^2(x_1^2 + x_2^2) + b^2(y_1^2 + y_2^2) + 2ax_1by_1 + 2ax_2by_2 \neq a^2(x_1^2 + x_2^2) + b^2(y_1^2 + y_2^2) = \sqrt{ax_1^2 + ax_2^2}^2 + \sqrt{by_1^2 + by_2^2}^2$. Extra crossover term.
- Remember $proj_v(\vec{x}) = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$
- So 1-form: $proj_v(x) \cdot \vec{u}$, because it's just $\vec{x} \cdot$ a bunch of stuff.
- However, GOTCHA, not a 1-form: $proj_v(x)$, since it outputs a vector, not a number!
- Obviously $proj_x(\vec{v}) = \frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \vec{x}$ is not, as both numerator and denominator are quadratic.
- An index into a vector is a one form: $[0, 1, 0] \cdot [x, yz] = y$
- A mean is a one form: $[\frac{1}{n}, \frac{1}{n}, \frac{1}{n}] \cdot [x, y, z]$
- Note: A zero-form is a function on $\vec{x}$, since it just returns a scalar at that point.

Define $dx_j[\hat{e}_i] = 1$ if $i = j$ else 0. Note: This is really a function $dx_j(\vec{x}) = [0, 0, 0, ...1, ...0] \cdot \vec{x}$, where the 1 is in the j spot. These are the **basis 1-forms** because $\boxed{\text{any 1-form can be written as } \omega = a_1 dx_1 + ... + a_n dx_n.}$. So $\omega[\vec{x}] = \vec{v} \cdot \vec{x}$ is just $\sum_{j=1}^n (v_j dx_j)[\vec{x}]$.

NOTE: This 1-form dx_2, say, looks an awful lot like dy, since $\vec{v} =< x, y, z >=< 0, y, 0 >\to dy[\vec{v}] = 1$

A **tensor** is a linear function to a number taking not just one but multiple vectors as inputs.
The **tensor product** is built from linear combinations of $(dx_i \otimes dx_j)(\vec{v}, \vec{w}) = dx_i(\vec{v}) \cdot dx_j(\vec{w}) = v_i w_j$, so $T(\vec{v}, \vec{w}) = (\sum_{i,j} t_{ij} dx_i \otimes dx_j)(\vec{v}, \vec{w}) = \sum_{i.j} t_{ij} v_i w_j$. Just combos of each pair. Examples and non-examples of 2-tensors:

- Tensor: $\vec{x} \cdot \vec{y}$. Linear, outputs a number.
- Non-Tensor: $\vec{x} + \vec{y}$. Linear, but outputs a vector.
- Non-Tensor: $\vec{x} \times \vec{y}$. Linear, but outputs a vector.
- Non-Tensor: $\|\vec{x}\| + \|\vec{y}\|$. Non-linear

As for 3-tensors:

- $\det(\vec{u},\vec{v},\vec{w})$ defined as $\det\begin{pmatrix} u_1 & v_1 & w_1 \\ u_1 & v_2 & w_2 \\ u_1 & v_3 & w_3 \end{pmatrix}$ is linear in changes to u, v, w.
- $(\vec{u}\times\vec{v})\cdot\vec{w}$ actually IS $\det\begin{pmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$
- $(\vec{u}\times\vec{v})\times\vec{w}$ outputs a vector.
- $(\vec{u}\cdot\vec{v})\vec{w}$ outputs a vector.
- A 3-tensor is generally $T(\vec{u},\vec{v},\vec{w}) = (\sum_{i,j,k} t_{ijk} dx_i \otimes dx_j \otimes dx_k)(\vec{i},\vec{v},\vec{w}) = \sum_{i.j,k} t_{ijm} u_i v_j w_k$, so think of it as $(dx_i \otimes dx_j \otimes dx_k)(\vec{u},\vec{v},\vec{w}) = u_i v_j w_k$

Side note: Wedge product $\vec{a}\wedge\vec{b} = \frac{1}{2}[\vec{a}\otimes\vec{b} - \vec{b}\otimes\vec{a}]$ is a square matrix measure of the anti-commutativity of $\vec{a}\otimes\vec{b}$. So, $(\vec{a}\wedge\vec{b})_{ij} = u_i v_j - u_j v_i$. (Note: Some texts seem to omit the $\frac{1}{2}$.) It has algebraically verifiable properties of:

- Antisymmetry: $(\vec{a}\wedge\vec{b}) = -(\vec{b}\wedge\vec{a})$.
  - This makes the tensor called **alternating**.
  - This also means $(\vec{a}\wedge\vec{a}) = -(\vec{a}\wedge\vec{a}) = [0]$
  - This also means (with basic associativity) that switching any two flips the sign: $\vec{a}\wedge\vec{b}\wedge\vec{c} = -\vec{b}\wedge\vec{a}\wedge\vec{a}$
- Bilinearity: $(c\vec{a}\wedge\vec{b}) = c(\vec{a}\wedge\vec{b})$
- Distributivity: $\vec{a}\wedge(\vec{b}+\vec{c}) = \vec{a}\wedge\vec{b} + \vec{a}\wedge\vec{c}$
- And others.

A **2-form** is an alternating 2-tensor: a linear operation taking in 2 vectors and producing a number, where switching the vectors flips the sign.

A **3-form** is an alternating 3-tensor: a linear operation taking in 3 vectors and producing a number, where flipping two vectors flips the sign, so "permutation rules" signage. So switch

A three dimensional wedge product continues this and again has all combinations :

$dx_i \wedge dx_j \wedge dx_k = \frac{1}{6}[$
$dx_i \otimes dx_j \otimes dx_k - dx_i \otimes dx_k \otimes dx_j$
$+ dx_j \otimes dx_k \otimes dx_i - dx_j \otimes dx_i \otimes dx_k$
$+ dx_k \otimes dx_i \otimes dx_j - dx_k \otimes dx_j \otimes dx_i]$

Form fields eat position $\vec{x}$ and return a form (tensor taking in vectors and returning a number).

- Main rule: $df = \sum_{j=1}^{n} \frac{\delta f}{\delta x_j} dx_j$.
- Implies somehow (?): $d(\sum_{j=1}^{n} f_j(\vec{x}) dx_j) = \sum_{j=1}^{n} [df_j] \wedge dx_j$.
- And suspending disbelief this means $d(\sum_{i,j=1}^{n} f_{ij}(\vec{x}) dx_i \wedge dx_j) = \sum_{i,j=1}^{n} [df_{ij}] \wedge dx_i \wedge dx_j$.
- Example: If $\omega = xdy - ydx = -ydx + xdy$
  - Set $f_1 = -y, f_2 = x$
  - $d\omega = df_1 \wedge dx + df_2 \wedge dy$ by second rule
  - $= (\frac{\delta f_1}{\delta x} dx + \frac{\delta f_1}{\delta y} dy) \wedge dx + (\frac{\delta f_2}{\delta x} dx + \frac{\delta f_2}{\delta y} dy) \wedge dy$ by main rule
  - $-dy \wedge dx + dx \wedge dy = 2dx \wedge dy$ by calcuation

Another example: $\omega = \frac{1}{2}(x^2 + y^2 + z^2)[dx \wedge dy + dx \wedge dz + dy \wedge dz]$ sees $d\omega$ as:

- $= [xdx + ydyzdz][dx \wedge dy + dx \wedge dz + dy \wedge dz]$ by third rule
- $= x(dx \wedge dy \wedge dz) + y(dy \wedge dx \wedge dz) + z(dz \wedge dx \wedge dy)$ since anything like $dx \wedge dx$ zeroes out
- $= (x - y + z) dx \wedge dy \wedge dz$

Manifolds are part of differential geometry, or really, just space curves (1-manifold), surfaces (2-manifolds), regions (3-manifolds). The number tells us how many numbers are needed to specify a point within it.

Rule: We can only integrate an n-form on an n-manifold. So, this means we can integrate a single (linear) vector-eating function on a line, a tensor eating two vectors on a surface, a 3-form on a region.

Be careful - it's not just the variable count: $\omega = xdx - ydz \to d\omega = (dx) \wedge dx - (dy) \wedge dz = -dy \wedge dz$. So it is a 2-manifold, and we can only integrate $d\omega$ on a surface.

Example: If we want to integrate $f(x, y, z) = x^2 + 2y^2 + 3z$ on $\vec{x}(t) = \langle 1 - t, t, 2 + 3t \rangle t \in [0, 1]$:

- $dx = \frac{dx}{dt} dt, dx = \frac{dy}{dt} dt, dz = \frac{dz}{dt} dt$
- $\int_M df = \int_M [2xdx + 4ydy + 6zdz]$
- $= \int_{t=0}^{1} [2x(t) \frac{dx}{dt} + 4y(t) \frac{dy}{dt} + 6z(t) \frac{dz}{dt}]$
- $= \int_{t=0}^{1} [2(1 - t)(-1) + 4(t)(1) + 6(2 + 3t)(3)] dt = 64$

Example: $M = x^2 + y^2 + z^2 \leq 1$, $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$. What's $\int_{\delta M} \omega$

- Parameterize
- $dx = \frac{\delta}{\delta u}(\sin(u)\cos(v)) + \frac{\delta}{\delta v}(\sin(u)\cos(v)) = \cos(u)\cos(v)du - \sin(u)\sin(v)dv$
- $dy = \frac{\delta}{\delta u}(\sin(u)\sin(v)) + \frac{\delta}{\delta v}(\sin(u)\sin(v)) = \cos(u)\sin(v)du + \sin(u)\cos(v)dv$
- $dz = \frac{\delta}{\delta u}(\cos(u)) = -\sin(u)du$
- $dx \wedge dy = (\cos(u)\cos(v)\sin(u)\cos(v) + \sin(u)\sin(v)\cos(u)\sin(v))du \wedge dv = \cos(u)\sin(u)du \wedge dv$
- $dx \wedge dz = -\sin(u)\sin(v)\sin(u)du \wedge dv = -\sin^2(u)\sin(v)du \wedge dv$
- $dy \wedge dz = -\sin(u)\sin(v)\sin(u)du \wedge dv = \sin^2(u)\cos(v)du \wedge dv$ (wrong should be negative)?
- $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$
- $= \sin(u)\cos(v)\sin^2(u)\cos(v)du \wedge dv - \sin(u)\sin(v)[-\sin^2(u)\sin(v)du \wedge dv] + \cos(u)\cos(u)\sin(u)du \wedge dv$
- $= [\sin^3(u) + \cos^2(u)\sin(u)]du \wedge dv$
- $= \sin(u)du \wedge dv$
- So $\int_{v=0}^{2\pi}\int_{u=0}^{u=\pi} \sin(u)dudv = 4\pi$

Example: Do it again by computing the other way $\int_M \delta\omega$:

- $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$
- $\delta\omega = 3dx \wedge dy \wedge dz$
- $\iint_M 3dxdydz = 3 \times \frac{4\pi}{3} = 4\pi.$
- **BIG**: This is one example of $\int_{\delta M} \omega = \int_M d\omega$

Note: The great unification happens when dumping different forms into $\int_{\delta M} = \int_M \delta\omega$

- If we use $\omega = F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy$, then we find that $d\omega = \nabla \cdot F dx \wedge dy \wedge dz$.
- Then $\int d\omega = \iiint_M \nabla \cdot F dxdydz$. (Why drop the wedges, exactly?)
- If we parametrize δM as $\vec{x}(u, v)$, with $dx = \frac{\delta x}{\delta u}du + \frac{\delta x}{\delta u}dv$, similar for the other two, and compute all the $dy \wedge dz$ etc....
- And we churn through noticing cross products inside, we get $\omega = \vec{F}(\vec{x}(u, v)) \cdot (\vec{u}_u \times \vec{x}_v)du \wedge dv$
- So $\int_{\delta M} \omega = \iint_D \vec{F}(\vec{x}(u, v)) \cdot (\vec{u}_u \times \vec{x}_v)du \wedge dv$

- $= \vec{F}(\vec{x}(u,v)) \cdot \frac{\vec{u}_u \times \vec{x}_v}{\|(\vec{u}_u \times \vec{x}_v)\|} \|(\vec{x}_u \times \vec{x}_v)\| du \wedge dv$
- $= \iint_{\delta M} \vec{F} \cdot d\vec{A}$
- So we've equated $\iiint_M \nabla \cdot F dxdydz = \iint_{\delta M} \vec{F} \cdot d\vec{A}$ after dropping the wedges somehow.
- Divergence Theorem!

Similarly, if we use $\omega = F_x dx + f_y dy + F_z dz$, we can do something similar to get $\int_{\delta M} \vec{F} \cdot d\vec{x} = \iint_M \nabla \times \vec{F} \cdot d\vec{A}$, or Stokes' Theorem.

NOTE: Still need some intuiition for these forms. Apparently these are described as "that which makes Stokes's and Divergence Theorem fall out": https://math.stackexchange.com/questions/2858098/is-a-differential-form

Note: From lecture at "https://www.youtube.com/watch?v=wlo2V8H5khM", $d_{i_1,i_2,...i_k}$ is a function that takes k vectors in $\mathbb{R}^n$, concatenates horizontally, selects only rows $i_1, i_2, ...i_k$ and takes the determinant. This is a multilinear, alternating function.

14 Chapter 5: Applications

14.1 5.1: The Laplacian

Laplacian: $\nabla^2 \vec{F} = \nabla \cdot \nabla \vec{F} = \frac{\delta^2}{\delta x^2} F_x + \frac{\delta^2}{\delta y^2} F_y + \frac{\delta^2}{\delta z^2} F_z$ shows up often in partial differential equations.

Example: Finding Potential V of an electrostatic field $\vec{E} = -\nabla V$, if $\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3}$

- Consider this in spherical coordinates and note that V is only a function of ρ.
- Use the gradient in spherical coordinates: $\nabla \cdot \vec{V} = \frac{\delta V}{\delta \rho}\hat{\rho} + \frac{1}{\rho}\frac{\delta V}{\delta \phi}\hat{\phi} + \frac{1}{\rho \sin(\phi)}\frac{\delta V_\theta}{\delta \theta}\hat{\theta}$, ignoring the second and third terms of the sum.
- $\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{x}}{\|\vec{x}\|^3} = \frac{Q}{4\pi\epsilon_0} \frac{1}{\rho^2}\hat{\rho} = -\nabla V = \frac{\delta V}{\delta \rho}\hat{\rho}$
- $\frac{Q}{4\pi\epsilon_0} \frac{1}{\rho^2} = -\frac{\delta V}{\delta \rho}$
- $\frac{Q}{4\pi\epsilon_0} \frac{1}{p} + C = V(\rho)$ or $\frac{Q}{4\pi\epsilon_0} \frac{1}{\|\vec{x}\|} = V(\vec{x})$ is potential for point charge at origin.
- This means a set of charges at points P_i have total potential $\sum_i \frac{Q}{4\pi\epsilon_0} \frac{1}{\|\vec{x} - \vec{P_i}\|}$, creating the field $\vec{E} = -\nabla V$
- So if $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ (Gauss's law), and $\vec{E} - \nabla V$, then:
- $\nabla \cdot \vec{E} = -\nabla \cdot (\nabla V) = -\frac{\delta}{\delta x}\frac{\delta V}{\delta x} - \frac{\delta}{\delta y}\frac{\delta V}{\delta y} - \frac{\delta}{\delta z}\frac{\delta V}{\delta z} = \boxed{= -\delta_x^2 V - \delta_y^2 V - \delta_z^2 = -\nabla^2 V}$

- Laplacian shorthand : $\nabla^2 = \nabla \cdot \nabla = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$

Example: Two infinite thin sheets of metal aligned with xy-plane. Function $T(z)$ is temperature, $T(0) = T_0, T(h) = 2T_0$. What is T?

- From Laplace, $\nabla^2 T = (\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2})T = \frac{\delta^2 T}{\delta z^2}$ since only z matters here.
- Anti-derive $T''(z) = 0$ twice to get $T(z) = Az + B$.
- $T(0) = T_0 \Rightarrow B = T_0 \Rightarrow T(z) = Az + T_0$
- $T(h) = 2T_0 \Rightarrow Ah = T_0 \Rightarrow A = \frac{T_0}{h} \Rightarrow T(z) = \frac{T_0}{h} z + T_0$

Exmple: Thin circlular disk of radius r floats on a cushion of air, over centered opening of radius ϵ. Pressure at opening is p_{in}, at edge (and outside) is necessarily p_{atm}. If also follows $\nabla^2 p(r) = 0$

- To get "nabla squared" (or cylindrical Laplacian equation) we want to get "nabla of a vector" (div) on "nabla of a function" (grad).
- (1) Cylindrical grad equation: $\nabla f = \frac{\delta f}{\delta r}\hat{r} + \frac{1}{r}\frac{\delta f}{\delta \theta}\hat{\theta} + \frac{\delta f}{\delta z}\hat{z}$
- (2) Cylindrical div equation: $\nabla \cdot \vec{F} = \frac{1}{r}\frac{\delta r F_r}{\delta r}\hat{r} + \frac{1}{r}\frac{\delta F_\theta}{\delta \theta}\hat{\theta} + \frac{\delta F_z}{\delta z}\hat{z}$
- Subbing (1) in to (2) yields $\boxed{\nabla \cdot (\nabla \vec{F}) = \frac{1}{r}\frac{\delta(r\frac{\delta f}{\delta r})}{\delta r}\hat{r} + \frac{1}{r}\frac{\delta}{\delta \theta}(\frac{1}{r}\frac{\delta f}{\delta \theta})\hat{\theta} + \frac{\delta^2 F f}{\delta z^2}\hat{z}} = 0$
- We know we don't depend on θ or z, so this is just $0 = \frac{1}{r}\frac{\delta(r\frac{\delta f}{\delta r})}{\delta r}$
- We know r is not 0, so multiply by r: $0 = \frac{\delta}{\delta r}(r\frac{\delta f}{\delta r}) \Rightarrow A = r\frac{\delta f}{\delta r} \Rightarrow \frac{A}{r} = \frac{\delta f}{\delta r}$
- Anti-derivatives yield $A\ln(r) + B = f(r)$, and specifically
  $A\ln(R) + B = p_{atm}$
  $Aln(\epsilon) + B = p_{in}$
- Subtract to get $A\ln(\frac{R}{\epsilon}) = p_{atm} - p_{in} = \Delta p \Rightarrow A = \frac{\Delta p}{\ln(\frac{R}{\epsilon})}$
- Sub our new A in to get $B = p_{in} - \ln(\epsilon)\frac{\Delta p}{\ln(\frac{R}{\epsilon})}$
- Subbing all gives us $p(r) = \Delta p \frac{\ln(\frac{r}{\epsilon})}{\ln(\frac{\epsilon}{R})} + p_{in}$

Challenge: If lift $L = \iint_{disk}[p - p_{atm}]dA = W$ weight causes the disk to float, what $\Delta p = p_{in} - p_{atm}$ satisfies it?

- We know form $r \in [\epsilon, R], p(r) = \Delta p \frac{\ln(\frac{r}{\epsilon})}{\ln(\frac{\epsilon}{R})} + p_{in}$
- And $r \in [0, \epsilon]$ means it is $p_i n$.

- Split the integral apart.
- $L = \int_{\theta=0}^{2\pi} \int_{r=0}^{r=R} p - p_{atm} dA$, or $L = 2\pi \int_{r=0}^{r=R} p(r)rdr - \pi * R^2 p_{atm}$ (term 1)
- $2\pi \int_{r=0}^{r=R} p(r)rdr = 2\pi(\int_{r=\epsilon}^{r=R} p(r)rdr) + 2\pi(\int_{r=0}^{r=\epsilon} p_{in}rdr)$. Second term is $\pi * \epsilon^2 p_{in}$
- $2\pi(\int_{r=\epsilon}^{r=R} \Delta p \frac{\ln(\frac{r}{\epsilon})}{\ln(\frac{\epsilon}{R})}) = \frac{2\pi\Delta p}{\ln(\frac{\epsilon}{R})} \int_{r=\epsilon}^{r=R} \ln(\frac{r}{\epsilon})rdr = \frac{2\pi\Delta p}{\ln(\frac{\epsilon}{R})} \int_{r=\epsilon}^{r=R} \ln(r)rdr - \frac{2\pi\Delta p \ln(\epsilon)}{\ln(\frac{\epsilon}{R})} * \frac{R^2-\epsilon^2}{2}$
- Last term above is term 3.
- Term 4 is $-\frac{R^2-\epsilon^2}{4} + \frac{R^2\ln(R)-\epsilon^2\ln(\epsilon)}{2}$
- $L = \frac{\Delta P}{\ln(\frac{R}{\epsilon})} \frac{\pi}{2}(R^2 - \epsilon^2) = W$
- $\Rightarrow \Delta p = \frac{2W\ln(\frac{R}{\epsilon})}{\pi(R^2-\epsilon^2)}$

Example: Capacitor for storing energy - Two concentric conducting shells, $R_1 < R_2$ voltage $V(R_1) = 0, V(R_2) = V_0$. No charge between the spheres so $\nabla^2 V = 0$.

- Use spherical coordinates. Everything obviously depends on ρ.
- (1) Spherical grad equation: $\nabla f = \frac{\delta f}{\delta\rho}\hat{\rho} + \frac{1}{\rho}\frac{\delta f}{\delta\phi}\hat{\phi} + \frac{1}{\rho\sin(\phi))}\frac{\delta f}{\delta\theta}\hat{\theta}$
- (2) Spherical div equation: $\nabla \cdot \vec{F} = \frac{1}{\rho^2}\frac{\delta(\rho^2 F_\rho)}{\delta\rho} + \frac{1}{\rho\sin(\phi)}\frac{\delta}{\delta\phi}(\sin(\phi)F_\phi) + \frac{1}{\rho\sin(\phi)}\frac{\delta F_\theta}{\delta\theta}$
- Subbing (1) in to (2) yields $\nabla^2 V = \frac{1}{\rho^2}\frac{\delta}{\delta\rho}[\rho^2\frac{\delta V}{\delta\rho}] + \frac{1}{\rho^2\sin(\theta)}\frac{\delta}{\delta\phi}(\sin(\phi)\frac{\delta V}{\delta\phi}) + \frac{1}{\rho^2\sin^2(\phi)}\frac{\delta^2 V}{\delta\theta^2}$
- Solving for the potential $V(\rho), R_1 \leq \rho \leq R_2$ is the same as above:
- $\frac{1}{\rho^2}\frac{\delta}{\delta\rho}[\rho^2\frac{\delta V}{\delta\rho}] = 0$. Multiply by ρ^2
- $A = \rho^2\frac{\delta V}{\delta\rho} \to A\rho^{-2} = \frac{\delta V}{\delta\rho} \Rightarrow B - A\rho^{-1} = V(\rho)$
- Churning through you get $B = \frac{A}{R_1}, A = \frac{V_0 R_1 R_2}{R_2 - R_1}$, and $V(\rho) = \frac{R_2 V_0}{R_2 - R_1}[1 - \frac{R_1}{\rho}]$
- And therefore the field $\vec{E} = -\nabla V$ (grad of potential) and Spherical grad equation (1) above mean that $\vec{E} = \frac{\delta V}{\delta\rho}\hat{\rho} = \frac{-R_1 r_2 V_0}{R_2 - R_1}\frac{\hat{\rho}}{\rho^2}$
- Note that this looks like the field from a point charge (proportional to $\frac{\vec{x}}{\|\vec{x}\|^3}$) for the same
- Given that spherical charge distribution vs. placed all at center is the same field (from previous chapters), this applies as well since they're spherical shells.

14.2 8.2:Gaussian Integrals I

Summary: A set of tricks for evaluating $I(a) = \int_{-\infty}^{\infty} x^n e^{-\frac{a}{2}x^2} dx$, specifically when $n = 0$: $(\sqrt{\frac{2\pi}{a}})$, even (recursive built on $n = 0$) or or odd (zero).

Gaussian for n=0:

- $I(a)$ is defined as a function of a with dummy variable x(or whatever): $I(a) = \int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2} dx$
- Trick to getting this integral is *integrating it in both directions* over the xy plane and taking the square root.
- Specifically, $I(a)^2 = (\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2} dx)(\int_{-\infty}^{\infty} e^{-\frac{a}{2}y^2} dy) = (\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{a}{2}(x^2+y^2)} dxdy$
- Rewrite in polar and solve: $I(a)^2 = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{\frac{a}{2}r^2} r dr d\theta = [2\pi * \frac{-1}{a} e^{-\frac{a}{2}r^2}]_0^{\infty} = \frac{2\pi}{a}$
- Thus $I(a) = \sqrt{\frac{2\pi}{a}}$

Gaussian for n=odd: $\int_{-\infty}^{\infty} x^n e^{-\frac{a}{2}x^2} dx, a > 0$: The $e^{f(x)}$ factor is symmetric, x^n is odd, so the whole thing is zero.

Gaussian for n=2:

- Use a derivative trick to *reproduce our tricky integral* from known identity $I(a) = \sqrt{\frac{2\pi}{a}}$ instead of trying to evaluate $\int_{-\infty}^{\infty} x^2 e^{-\frac{a}{2}x^2} dx)$
- $\frac{\delta}{\delta a} I(a) = \frac{\delta}{\delta a} \sqrt{2\pi} a^{-\frac{1}{2}} = -\frac{1}{2}\sqrt{2\pi} a^{-\frac{3}{2}}$
- $\frac{\delta}{\delta a} I(a) = \int_{-\infty}^{\infty} \frac{\delta}{\delta a} [e^{-\frac{a}{2}x^2}] dx = -\frac{1}{2} \int_{-\infty}^{\infty} x^2 [e^{-\frac{a}{2}x^2}] dx$
- Equating the ends of the last two lines shows that $\int_{-\infty}^{\infty} x^2 [e^{-\frac{a}{2}x^2}] dx = \sqrt{2\pi} a^{-\frac{3}{2}}$
- From here, we can attack $n = 4$, then from there, $n = 6$, etc.

If we're going to evaluate the general form $\int_{-\infty}^{\infty} [e^{-\frac{a}{2}x^2 - bx + c}] dx$:

- First, complete the square of $f = -\frac{a}{2}x^2 - bx + c$

$$-\frac{2}{a} f = x^2 + \frac{2b}{a} x - \frac{2c}{a} \tag{96}$$

$$-\frac{2}{a} f = (x + \frac{b}{a})^2 - \frac{b^2}{a^2} - \frac{2c}{a} \tag{97}$$

$$f = -\frac{a}{2}(x + \frac{b}{a})^2 + \frac{b^2}{2a} + c \tag{98}$$

$$\tag{99}$$

Therefore, $\int_{-\infty}^{\infty}[e^{-\frac{a}{2}x^2-bx+c}]dx = \int_{-\infty}^{\infty}[e^{-\frac{a}{2}(x+\frac{b}{a})^2+\frac{b^2}{2a}+c}]dx = [e^{\frac{b^2}{2a}+c}]\int_{-\infty}^{\infty}[e^{-\frac{a}{2}(x+\frac{b}{a})^2}]dx$. Notice that $u = x + \frac{b}{a}, du = dx$ makes the integral $\sqrt{\frac{2\pi}{a}}$.

So general form of Gaussian, with $n = 0$, is $\boxed{\int_{-\infty}^{\infty}[e^{-\frac{a}{2}x^2-bx+c}]dx = \sqrt{\frac{2\pi}{a}}[e^{\frac{b^2}{2a}+c}]}$.

14.3 8.2:Gaussian Integrals I—

Summary: Multivariable Gaussian integrals are quadratic combos of x, y, z, etc. and $\boxed{\int_{\mathbb{R}^2} e^{-\frac{1}{2}[\vec{x}\cdot(A\vec{x})]} = \sqrt{\frac{(2\pi)^n}{\det(A)}}}$. To evaluate them, you can split them out Fubini-style and multiply, you can get to a diagonal matrix, or you can rotate to a diagonal and back, which has the same determinant.

Steps:

- If A $= aI_{n\times n}$, then $\int_{\mathbb{R}^2} e^{-\frac{1}{2}[\vec{x}\cdot(A\vec{x})]} = (\sqrt{\frac{2\pi}{a}})^n$,
  - since $\int e^{-\frac{1}{2}[ax^2+ay^2...]}dxdy = \int e^{-\frac{1}{2}[ax^2]}dx \int e^{-\frac{1}{2}[ay^2]}dy$
- If A is a diagonal matrix with entries $\lambda_1\lambda_2...$, then $\int_{\mathbb{R}^2} e^{-\frac{1}{2}[\vec{x}\cdot(A\vec{x})]} = \sqrt{\frac{(2\pi)^n}{\lambda_1\times\lambda_2...}}$
  - This is the same Fubini setup as above. Separate the integrals and multiply results.
- If A is a symmetric matrix, $\int_{\mathbb{R}^2} e^{-\frac{1}{2}[\vec{x}\cdot(A\vec{x})]} = \sqrt{\frac{(2\pi)^n}{\det(A)}}$
  - We can take any symmetric matrix and change coordinates to get a diagonal matrix (TODO: I forget why).
  - Then $A = RDR^T$, and since R is a rotation matrix $\det(R) = \det(R^T) = 1$

Example: Evaluate $\frac{1}{2\pi}\iint_{\mathbb{R}^2} \exp[-\frac{1}{2}[2x^2 - 2xy + 5y^2]]dxdy$

- A matrix $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, and writing $\vec{x} = \langle x, y\rangle$, together presented as $\vec{x}\cdot(A\vec{x})$, means that $a = 2, b = -1, d = 5$ by just expanding the multiplication explicitly.
- since $\det(A) = 9$, then $\sqrt{\frac{(2\pi)^2}{9}} = 2\pi\frac{1}{3}$

Example: Evaluate $\frac{1}{2\pi}\iint_{\mathbb{R}^2} \exp[-\frac{1}{2}[2x^2 + y^2 - 8x - 2y + 9]]dxdy$

- Rewrite the exponent as $[-\frac{1}{2}[2(x-2)^2 + (y-1)^2]$
- Separate out into two integrals $\int f(x)dx \int g(y)dy$

- The dx integral is of the form $\int \exp[-\frac{a}{2}u^2]$, with $u = x - 2$. So that's $\sqrt{\frac{(2\pi)}{2}}$
- The dy integral is of the form $\int \exp[-\frac{a}{2}v^2]$, with $v = y - 1$. So that's $\sqrt{\frac{(2\pi)}{1}}$
- Multiplying together, you get $\frac{2\pi}{\sqrt{2}}$

Example: Evaluate $\frac{1}{2\pi}\iint_{\mathbb{R}^2} \exp[-\frac{1}{2}[-2x^2 + 2xy - 2y^2 - 6x + 6y - 6]]dxdy$. This is mixed so trickier,.

- Rewrite in the format $\exp[-\frac{1}{2}\vec{x} \cdot (A\vec{x}) + \vec{b} \cdot \vec{x} + c$.
- Here, $A = \begin{pmatrix} 2 & -1 \\ -1 & 2d \end{pmatrix}, \vec{b} = \langle -3, 3 \rangle, c = -3$
- Linear algebra says we can transform to a diagonal matrix : eigenvectors rotate to the eigenspace, then expand via the diagonal, then rotate back)
- Once you get these R and R^T matrices, you can find the change of coordinates $u = \frac{1}{\sqrt{2}}(x + y), v = frac1\sqrt{2}(-x + y)$
- $\Rightarrow x = \frac{1}{\sqrt{2}}[u + v], y = \frac{1}{\sqrt{2}}[u - v]$
- NOTE: They then mention the Jacobian J where $|\det \begin{pmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{pmatrix}| = 1$
- This leaves us with$\frac{1}{2\pi}\iint_{\mathbb{R}^2} \exp[-\frac{1}{2}[u^2 + 3v^2] + 3\sqrt{2}v - 3]dudv$
- Rewrite exponent to $-\frac{1}{2}u^2 - \frac{3}{2}(v - \sqrt{2})^2$ to get $\frac{2\pi}{\sqrt{1}\sqrt{3}} \times 2\pi = \frac{1}{\sqrt{3}}$
-

14.4 8.4: Fourier Transforms

Summary: By employing the Dirac Delta function $\boxed{\delta(\vec{x}) \approx \int_{\mathbb{R}^n} e^{2\pi i\vec{x}\cdot\vec{k}} d\vec{k}}$, we can transform functions into a separate eigenspace where derivatives and integrals may be easier.

Note from the internet: The Fourier transform maps like a $3\sin(x)$ waveform to a function which has a peak at (frequency) 3. This makes sense that the Dirac ends up being a bunch of "point spikes". More to figure out.

- Motivation: While linear transform $A = RDR^T$ transforms e_i to eigenvectors w_i: $Re_i \to w_i$, where A is diagonal, so that $Aw_i = \lambda_i w_i$
- Similarly, operators like ∇ change functions into vector-valued functions. So an "eigenfunction" of ∇ would have $\nabla f_{\vec{k}}(\vec{x}) \propto k f_{\vec{k}}(\vec{x})$

- The function $e^{2\pi i\vec{k}\cdot\vec{x}}$ fits that bill.
- So in the same way that the eigenvector basis w_i means any $\vec{v} = \sum a_j \hat{w}_j$, $\boxed{\text{any function } f(\vec{x}) = \int_{\mathbb{R}^n} \hat{f}(\vec{k}) e^{2\pi i\vec{k}\cdot\vec{x}} d\vec{k} \text{ for some function } \hat{f}_{\vec{k}} \text{ called the } \textbf{Fourier Transform} \text{ of } f}$.
- This looks like a pretty crappy integral to evaluate, so we'll learn some tricks here.

Things to remember about $\delta(x)$

- $\delta(x) = 0$ except at $x = 0$. Another way to think about this is $\delta(x-a) = 0$ except at $x = a$.
- This also means $f(x)\delta(x-a) = f(a)\delta(x-a)$ since $f(x)$ only comes into play when $x = a$. *Note: This is useful for getting rid of x in some integrals*
- $\int_{\mathbb{R}} \delta(x-a) = 1$ for any a.
- Can we approximate δ using normal functions?
  - This basically looks like a Gaussian $\propto e^{\frac{-x^2}{2}}$ with an extremely sharp center.
  - Normalizing: If we set $A = \frac{1}{\sqrt{2\pi\epsilon}}$, then $\int_{\mathbb{R}} Ae^{\frac{-x^2}{2}} dx = 1$.
  - If we take the limit as $\epsilon \to 0$, then $\delta(x) \approx \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} \approx \int_{\mathbb{R}} e^{-2\pi^2\epsilon k^2} e^{2\pi i k x} dk$ if ϵ is small.
  - Subbing $\epsilon = 0$, which is not technically allowed, makes this look like $\int_{\mathbb{R}} e^{2\pi i k x} dk$
  - This generalizes as $\delta(\vec{x} - \vec{a}) = \int_{\mathbb{R}^n} e^{2\pi i\vec{k}\cdot[\vec{x}-\vec{a}]} d\vec{k}$
- So above we've basically posited the transformation $f(\vec{x}) = \int_{\mathbb{R}^n} \hat{f}(\vec{k}) e^{2\pi i\vec{x}\cdot\vec{k}} d\vec{k}$ from Fourier space to regular space.
- This means that $\hat{f}(\vec{k}) = \int_{\mathbb{R}^n} f(\vec{x}) e^{-2\pi i\vec{k}\cdot\vec{x}} d\vec{x}$ works the other way. How to rip out $\hat{f}(\vec{k})$ from the integral?
  - We know $\int \delta(k'-k)dk' = 1$, so $\hat{f}(k) = \hat{f}(k) \int \delta(k'-k)dk'$
  - $= \int \hat{f}(k)\delta(k'-k)dk'$ by moving a k-based function inside the k'-based integral.
  - $= \int \hat{f}(k')\delta(k'-k)dk'$ by the delta function rule.
  - $= \int \hat{f}(k')[\int e^{2\pi i x[k'-k]}]dk'$ by Gaussian-based definition of δ.
  - $= e^{2\pi i x[-k]} \int \hat{f}(k') e^{2\pi i x k'} dk'$ by pulling out the k terms from the k' integral.
  - $= f(x)e^{-2\pi i x k}$ by definition of $f(x)$ from above.

- So **transform to Fourier space** is $\hat{f}(\vec{k}) = \int_{\mathbb{R}^n} f(\vec{x}) e^{-2\pi i \vec{k}\cdot\vec{x}} d\vec{k}$
- Playing around in Fourier space:
  - $\hat{\frac{df}{dx}}(k)$, meaning the transform of derivative $f'(x)$, is $2\pi i k \hat{f}(k)$ after integrating $\int_{\mathbb{R}} f'(x) e^{-2\pi i x\cdot k} dx$ by parts, with $dv = f'(x), u = e^{-2\pi i x\cdot k}$.
  - Since all the components are independent, the two-variable case $[2\pi i]^2 k_i k_j \hat{f}(\vec{k}) = \frac{\delta^2 \hat{f}}{\delta x_i \delta x_j}$ follows.

Example of solving diff eqs with Fourier Transform:

- Applying this, the differential equation $\frac{d^2 f}{dx} + \omega^2 f = 0$ admits the solution $([2\pi i k]^2 + \omega^2]\hat{f}(\vec{k}) = 0$
- If we assume $\hat{f}(\vec{k}) \neq 0$ *everywhere*, then it has values only at $k = \pm\frac{\omega}{2\pi}$
- This suggests the general form $a\delta(k - \frac{\omega}{2\pi}) + b\delta(k + \frac{\omega}{2\pi})$
- Let's evaluate one of the fourier terms: $g(x) = a\int \delta(k - \frac{\omega}{2\pi}) e^{2\pi k x} dk$
- Consider $f(x) = e^{2\pi k x}, a = \frac{\omega}{2\pi}$ and use the delta shift rule.
- So the integral ends up being $f(x) = f(a) = a e^{2\pi i (\frac{\omega}{2\pi})\cdot x} \int \delta(k - \frac{\omega}{2\pi}) dk = a e^{i\omega x}$
- The "b" term evaluates to $be^{-i\omega x}$
- Writing $f(x) = ae^{i\omega x} + be^{-i\omega x}$ using only real quantities: sub in $e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$.
- This yields the general solution $[a+b]\cos(\omega x) + i[a-b]\sin(\omega x)$. Define $A = a+b, B = a-b$ for general solution $f(x) = A\cos(\omega x) + B\sin(\omega x)$

14.5 5.5: Diffusion Equation

Laplacian: $0 = \nabla^2 T = \frac{\delta^2 T}{deltax^2} + \frac{\delta^2 T}{deltay^2} + \frac{\delta^2 T}{deltaz^2}$ looks similar but not the same.

Derivation of Diffusion Equatin.

- Diffusion: $\frac{\delta u}{deltat} = D\nabla^2 u$
- Definitions:
  - $D > 0$ is a **diffusion constant**, different per material
  - $u(\vec{x}, t)$ is unit density function - probability of seeing particle in some point in space. Integrates over $\mathbb{R}^3$ to 1

- $\rho(\vec{x},t) = Nu(\vec{x},t)$ is overall density function - N number of particles times the unit density function.
- $\vec{J}$ is the **current density**. Magnitude is particles cross a unit area per unit time. Direction points in average particle motion.
- Therefore $\iint_S \vec{J}\cdot d\vec{A}$ is the rate at which particles enter and exit the whole surface S. *Just like the flux.*
- Therefore $-\iint_S \vec{J}\cdot d\vec{A} = \iiint_V \frac{\delta\rho}{\delta t}(\vec{x},t)d\vec{x}$, since the flux is the particles leaving, and ρ is the particle (remaining) density.
- But by the divergence theorem, $-\iint_S \vec{J}\cdot d\vec{A} = \iiint_V \nabla\cdot\vec{J}(\vec{x},t)d\vec{x}$
- Equating these triple integrals, $\frac{\delta\rho}{\delta t} = \nabla\cdot\vec{J}(\vec{x},t) \Rightarrow \frac{\delta\rho}{\delta t} + \nabla\vec{J} = 0$
- Also, $\boxed{\text{Fick's Law: } \vec{J} = -D\nabla\rho}$, since the rate at which we're leaving an area is the negative of the gradient (which points towards the quantity increasing).
- Therefore, $\frac{\delta\rho}{\delta t} = D\nabla^2\rho \Rightarrow \frac{\delta u}{\delta t} = D\nabla^2 u$ since $\rho = Nu$.

Now we have to get to some Fourier magic to solve this equation.

- Start from $\frac{\delta u}{\delta t}$ on the one side.
- $\frac{\delta u}{\delta t} = D\nabla^2 u$
- $u = \iiint_{\mathbb{R}^3} \hat{u}(\vec{k},t)e^{2\pi i\vec{x}\vec{k}}d\vec{k}$ (Fourier OUT)
- $\frac{\delta u}{\delta t} = D(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2})\iiint_{\mathbb{R}^3} \hat{u}(\vec{k},t)e^{2\pi i\vec{x}\vec{k}}d\vec{k}$
- $= \iiint_{\mathbb{R}^3}[2\pi i]^2(\vec{k}\cdot\vec{k})\hat{u}(\vec{k},t)e^{2\pi i\vec{x}\vec{k}}d\vec{k} = \iiint_{\mathbb{R}^3}[-4\pi^2\|\vec{k}\|]^2\hat{u}(\vec{k},t)e^{2\pi i\vec{x}\vec{k}}d\vec{k}$
- But form the other side and the definition of Fourier OUT: $\frac{\delta u}{\delta t} = \iiint \frac{\delta u}{\delta t}e^{2\pi i\vec{x}\vec{k}}d\vec{k}$
- Therefore, by equating the contents of the integrals, we find the derivative on the Fourier side: $\frac{\delta\hat{u}}{\delta t} = D\cdot -4\pi^2\|k\|^2\hat{u}$
- Therefore, we can use the standard $f' = kf$, and the knowledge that $t = 0 \rightarrow \hat{u} = \hat{u}(\vec{k},0)$ to find **the solved diff eq on the Fourier side** $\boxed{\hat{u}(\vec{k},t) = \hat{u}(\vec{k},0)e^{-4\pi^2\|k\|^2 t}}$ if we assume $D = 1$

Then it looks like we have what appears to be a standard dance:

- Put this $\hat{u}$ into the Fourier OUT equation.
- Assume a FOURIER IN of some y so that $\hat{u}(\vec{k},0) = \iiint u(\vec{y},0)e^{-2\pi i\vec{y}\cdot\vec{k}}d\vec{y}$

- Rewrite so to shift $\vec{y}$ into the $d\vec{k}$ part
- Complete the square in the exponent and evaluate.
- Leading us to $\boxed{u(\vec{x},t) = (\frac{1}{4\pi Dt})^{\frac{3}{2}} \iiint u(\vec{y},0) e^{-\frac{1}{4Dt}\|x-y\|^2} d\vec{y}}$ as the almost full solution. Just need the initial constant.
- Note that $u(\vec{y},0)$ is:
  * The initial probability distribution. The particle is at $\vec{a}$, so it's "infinitely likely" there, and 0 elsewhere.
  * Therefore $u(\vec{y},0) = \delta x - a$. So we only need to evaluate at $\vec{y} = \vec{a}$.
  * So, since $\delta y - a$ confers ALL the distribution when $y = a$, then $(\frac{1}{4\pi Dt})^{\frac{3}{2}} \int_{\mathbb{R}^3} \delta(y - a) e^{-\frac{1}{4Dt}\|x-y\|^2} d\vec{y} = (\frac{1}{4\pi Dt})^{\frac{3}{2}} e^{-\frac{1}{4Dt}\|x-a\|^2}$f
  * **This trick** looks useful! Taking the fourier transform of a delta with a function (taking care to keep the $d\vec{x}$ or $d\vec{k}$ out of it, looks like it yields the function!
  *
-

Example: How far does the average particle travel, if $u(x,t) = (\frac{1}{4\pi Dt})^{\frac{3}{2}} \iiint_{\mathbb{R}^3} e^{-\frac{1}{4DT}\|x-a\|^2}$

- We're looking to estimate average distance, or $\sqrt{\iiint_{\mathbb{R}^3} \|x-a\|^2 u(\vec{x},t) d\vec{x}}$
- Consider $\vec{a} = 0$ (move to origin)
- Let's apply square root at the end.
- Switch to polar coordinates with (see above) $d\vec{x} = \rho^2 \sin(\phi) d\rho d\phi d\theta$
- THen we're looking at constant factors $(\frac{1}{4\pi Dt})^{\frac{3}{2}}$, 2 from integrating $\sin(\phi)$ from $[-\pi, \pi]$, and 2π from integrating θ.
- The integral becomes $\int \rho^4 e^{-\frac{1}{4\pi Dt}\rho^2} d\rho$
- Set $u = \frac{\rho}{2\sqrt{Dt}}, du = \frac{d\rho}{2\sqrt{Dt}}$
- Integral becomes $(2\sqrt{Dt})^4 (2\sqrt{Dt}) \int u^4 e^{-u^2} du = (2\sqrt{Dt})^4 (2\sqrt{Dt}) \frac{3\sqrt{\pi}}{8} * (\frac{1}{4\pi Dt})^{\frac{3}{2}} * 2 * 2\pi$
- $= 6Dt$. So result is the square root.
- Therefore, if $D = 6$, say, a particle will travel 1200 cm in on average $\sqrt{6 * 6 * t} = 1200 \Rightarrow t = 400$ seconds.

14.6 5.6: The Wave Equation

Summary: TODO

Derivation of Wave Equation:

- Fundamental idea: line of particles spaced l length apart, labeled by initial position $x + kl$, with *displacements* from these positions $u(x - l, t), u(x, t), u(x + l, t)$.
- So, $u(x+l, t) < u(x, t)$, for example, if the right-hand ball is a little left of start, and the left-hand ball a little right of start.
- Force law: $F = k|l' - l|$.
- Therefore, length of spring between right and middle: $u(x + l, t) + l - u(x, t)$, with rightward force $k|u(x + l, t) + l - u(x, t) - l| = k|u(x + l, t) - u(x, t)|$
- Therefore, length of spring between middle and left: $u(x, t) + l - u(x - l, t)$, with leftward force $-k|u(x + l, t) + l - u(x, t) - l| = -k|u(x, t) - u(x - l, t)|$
- Add these together to get $k|u(x+l, t) + u(x-l, t) - 2u(x, t)|$. Note that this is messy but all cases apply symmetrically.
- To estimate this, *use Taylor approximation*
  - Expand around center x: $T(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x - x_0)^2 + ...$
  - $T(x + l) = u(x + l, t) \approx u(x, t) + u'(x, t)l + \frac{1}{2}u''(x, t)l^2 + ...,$
  - $u(x - l, t) \approx u(x, t) + u'(x, t)(-l) + \frac{1}{2}u''(x, t)(-l)^2 + ...$
  - $\Rightarrow u(x + l, t) + u(x - l, t) - 2u(x, t) \approx u''(x, t)l^2$
  - So total force is then $kl^2 \frac{\delta^2 u}{\delta x^2}$
  - With $F = ma$, this means $F = m\frac{\delta^2 u}{\delta t^2} = kl^2 \frac{\delta^2 u}{\delta x^2} \Rightarrow \frac{\delta^2 u}{\delta t^2} = \frac{kl^2}{m}\frac{\delta^2 u}{\delta x^2}$
  - With "wave speed" v, for some reason $v^2 = \frac{kl^2}{m}$, for **general wave equation form** $\boxed{\frac{\delta^2 u}{\delta t^2} = v^2\nabla^2 u}$
  - Note: For rest of chapter, we set $v = 1$ (for simplicity) and add damping term for energy loss, so $\frac{\delta^2 u}{\delta t^2} + 2\gamma\frac{\delta u}{\delta t} = v^2\nabla^2 u$

Note: The sides of the drum are fixed on a drumhead of $[0, l_x] \times [0, l_y]$, so $u(x, 0, t) = u(0, y, t) = u(x, l_y, t) = u(l_x, y, t) = 0$

How can $\hat{u}(k,t)$ guarantee periodicity in $u(x,t)$? If $\boxed{\hat{u}(k,t) = \hat{u}(k,t)e^{2\pi ikl}}$, then

$$u(x+l,t) = \int \hat{u}(k,t)e^{2\pi ik(x+l)k}dk \tag{100}$$

$$= \int (\hat{u}(k,t)e^{2\pi ikl})e^{2\pi ikx}dk \tag{101}$$

$$= \int \hat{u}(k,t)e^{2\pi ikx} = u(x,t) \tag{102}$$

- TODO