Vandermonde Determinants via Graph Tournaments

Dave Fetterman

Obviously Unemployed

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Abstract

In 2D space, two points $(x_1, y_1), (x_2, y_2), x_1 \neq x_2$ define a line, a polynomial of degree 1. Three distinct points $(x_1, y_1), (x_2, y_2), (x_3, y_3), x_1 \neq x_2 \neq x_3$ define a parabola, a polynomial of degree 2. In general, for finite univariate polynomials of nonnegative, whole degree, $n + 1$ such points uniquely specify a polynomial of degree n. Why?

This is the farthest thing from a new result. This is a paper is instead a thoroughly awkward trip through a few mathematical domains including linear algebra and graph theory to arrive at this well known destination. Helicopters and cars both have their uses. But you wouldn't build a car by turning a helicopter on its side and adding wheels. Metaphorically, I do, so you don't have to.

1 Motivator: Polynomial Uniqueness in Interpolation

If we have points $f(x_0) = y_0, f(x_1) = y_1, \ldots f(x_n) = y_n$, how can we determine the coefficients a_i of the polynomial $f(x) = a_0x^0 + a_1x^1 + \ldots + a_nx^n$?

This square matrix of width $n + 1$, which I'll denote X_n, is known as a Vandermonde matrix[**1**], and models this set of $n + 1$ equations as $X \cdot \vec{a} = \vec{y}$:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Therefore, we can find our unique coefficient vector $\vec{a}$ if and only if we can solve $X_n \cdot \vec{a} = \vec{y}$, or $\vec{a} = X_n^{-1}\vec{y}$. $\vec{a}$ has a unique solution if and only if $\det(X_n) \neq 0$.

The rest of this paper tries to find this determinant through a uniquely circuitous path.

2 Finding the Vandermonde determinant

It should be noted that there are other, clearer methods of finding this determinant[**1**] either starting with polynomial unqiueness (basically, going the "other" direction), abstract algebra, direct linear algebra, vector maps, and others. These, however, were not the ones I stumbled on.

First, we know that for in any X_n, if any $x_i = x_j$ for distinct i, j, we have a zero determinant and no solution. If $f(x_i) = f(x_j)$ and $x_i = x_j$, then we are simply underdetermined (not enough points for a unique polynomial). If $f(x_i) = f(x_j)$ and $x_i \neq x_j$, then we have an impossible vertical section of our graph. Otherwise, we are in good shape to solve the above equation for $\vec{a}$.

This suggests that every pair $(x_i, x_j), i < j$ corresponds to a factor $(x_j - x_i)$ in the determinant, and that the determinant is then some multiple of $D = \prod_{0 \leq i < j \leq n}(x_j - x_i)$.

Taking $n = 2$ as a base case ($n = 1$ produces a boring constant $f(x)$), we see that $\det \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} = (x_1 - x_0)$, suggesting D may be the determinant of a Vandermonde matrix.

Let's prove that it is.

2.1 Setup: Vandermonde inductive step and main theorem

Theorem: The determinant of X_n with generating coefficients $x_0, x_1 ... x_n$ is $\prod_{0 \leq i < j \leq n}(x_j - x_i)$.

With the base case $n = 1$ in hand, the rest of the paper handles the inductive step of proving the main theorem. We assume in the inductive step that this determinant holds for Vandermonde matrices of size $n - 1$ ($n - 1$ generating coefficients, matrix with width n).

Inductive Step of Proof of Theorem:

If, for all X_{n-1}, $\det(X_{n-1}) = \prod_{0 \leq i < j \leq n-1}(x_j - x_i)$, then for all $X_n, \det(X_n) = \prod_{0 \leq i < j \leq n}(x_j - x_i)$.

2.1.1 Definitions

Let's create a few definitions:

- Denote by $M_{k,n}$ the Vandermonde matrix X_n with column n and row k excluded[1] , often called a "matrix minor".
- Given an ordered set of indices $I = [0, n] \in \mathbb{N}$, denote by P_I the *product* of all factors the form $(x_j - x_i)$, given $i < j$ and $i, j \in I$. So $P_{[0,2]} = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1)$.
- Given an ordered set of indices $I = [0, n] \in \mathbb{N}$, denote by S_I the *sum* over all permutations[2] σ of I of $sgn(\sigma)x^n_{\sigma(n)}x^{n-1}_{\sigma(n-1)}...x^0_{\sigma(0)}$. So $S_{[0,2]} = x^2_2x^1_1x^0_0 - x^2_2x^1_0x^0_1 - x^2_1x^1_2x^0_0 + x^2_1x^1_0x^0_2 + x^2_0x^1_2x^0_1 - x^2_0x^1_1x^0_2$.

2.1.2 Plan for Inductive Step of Vandermonde Derminant Proof

Using the definitions of P_I, S_I from above, the main thrust of the paper is proving this algebraic statement:

The P-S Equivalence Lemma: For a set of indices I, $P_I = S_I$.

This is the main statement we prove through the paper.

Assuming this is proven, the rest of the proof of the inductive step falls out quickly. We'll get that out of the way.

- (1) $det(X_n) = \sum_{k=0}^{n}(-1)^{k+n}x^n_k \det(M_{k,n})$
- (2) For our base base, $\det(\mathbf{X_1}) = P_{[0,1]}$
- (3) By inductive hypothesis $det(M_{k,n}) = \sum_{k=0}^{n}(-1)^{k+n}x^n_k P_{[0,n]-\{k\}}$
- (4) $\sum_{k=0}^{n}(-1)^{k+n}x^n_k S_{[0,n]-\{k\}} = S_{[0,n]}$
- (5) By the Lemma, (4) means $\sum_{k=0}^{n}(-1)^{k+n}x^n_k P_{[0,n]-\{k\}} = P_{[0,n]}$
- (6) Therefore, transitively, $det(X_n) = P_{[0,n]} = \prod_{0\leq i<j\leq n}(x_j - x_i)$.

2.1.3 Straightforward Steps (1)-(6) in Plan

(1) is the minor-based definition of the determinant.

The determinant of $X = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix}$ can be calculated down the rightmost column as

[1]Note: I use zero-indexed matrices in this paper, since in the case of a Vandermonde matrix X_n, the zero-indexed entry (i, j) neatly corresponds to x_i^j

[2]meaning, $\sigma \in Sym(I)$

$\det(X) = (-1)^n[x_0^n \det(M_{0,n}) - x_1^n \det(M_{1,n}) + ... + (-1)^n x^n \det(M_{n,n})]$.

(2) is clear, with $\det(\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix}) = -1 \cdot (1 \cdot M_{1,1} - 1 \cdot M_{0,1}) = (x_1 - x_0) = P_{[0,1]}$.

(3) says inductively, we can presuppose that for any $M_{k,n}$, *which is itself a Vandermonde matrix of smaller size*, $\det(M_{k,n})$ can be expressed as $P_{[0,n]-\{k\}}$

(4) This simply partitions all terms of the form $sgn(\sigma)x^n_{\sigma(n)}x^{n-1}_{\sigma(n-1)}...x^0_{\sigma(0)}$, $\sigma \in Sym(I)$ into those that start with x_k^n and no x_k in the tail, summed over all k. On $I = \{c, b, a\}$, for example, the terms split out exactly into $c^2(b^1a^0 - b^0a^1) - b^2(c^1a^0 - a^1c^0) + a^2(c^1b^0 - b^1c^0) = c^2b^1a^0 - c^2a^0b^1 - b^2c^1a^0 + b^2a^1c^0 + a^2c^1b^0 - a^2b^1c^0$.

(5) follows from applying the P-S equivalence Lemma to swap instances of S with those of P in (4).

(6) Following the equalities all the way back to 1, $\det(X_n)$ is then $P[0, n]$.

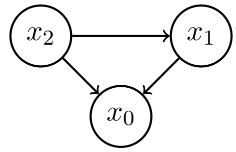
3 Proof of $P_{[0,n]} = S_{[0,n]}$

First, let's establish some algebraic intuition with an example.

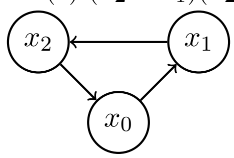
3.1 Example: $P[0, 1] = S[0, 1] \Rightarrow P[0, 2] = S[0, 2]$

We've already established that $P_{[0,1]} = (x_1 - x_0) = x_1^1x_0^0 - x_0^1x_1^0 = S_{[0,1]}$. To see that $P_{[0,2]} = S_{[0,2]}$, write it out:

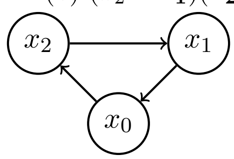
$$
\begin{aligned}
& P_{[0,2]} && (1) \\
&= (x_2 - x_1)(x_2 - x_0)P_{[0,1]} && (2) \\
&= (x_2 - x_1)(x_2 - x_0)S_{[0,1]} && (3) \\
&= x_2^2(x_1^1x_0^0 - x_0^1x_1^0) && (4) \\
&\quad -x_2x_0(x_1^1x_0^0 - x_0^1x_1^0) && (5) \\
&\quad -x_2x_1(x_1^1x_0^0 - x_0^1x_1^0) && (6) \\
&\quad +x_1x_0(x_1^1x_0^0 - x_0^1x_1^0) && (7) \\
&= x_2^2x_1^1x_0^0 + x_1^2x_0^1x_2^0 + x_0^2x_2^1x_1^0 && (8) \\
&\quad -x_2^2x_0^1x_1^0 - x_1^2x_2^1x_0^0 - x_0^2x_1^1x_2^0 && (9) \\
&\quad +x_0^1x_1^1x_2^1 - x_0^1x_1^1x_2^1 && (10) \\
&= S_{[0,2]} && (11)
\end{aligned}
$$



(a) $(\mathbf{x_2} - x_1)(\mathbf{x_2} - x_0)(\mathbf{x_1} - x_0) \rightarrow x_2^2 x_1^1 x_0^0$



(b) $(x_2 - \mathbf{x_1})(\mathbf{x_2} - x_0)(x_1 - \mathbf{x_0}) \rightarrow x_2^1 x_1^1 x_0^1$



(c) $(\mathbf{x_2} - x_1)(x_2 - \mathbf{x_0})(\mathbf{x_1} - x_0) \rightarrow -x_2^1 x_1^1 x_0^1$

Figure 1: Three terms of $P_{[0,2]}$, corresponding to complete directed graphs of size 3

Note that line (3) follows from line (2) by base case, line (8) is the set of even (subscript) permutations of $\{2, 1, 0\}$, line (9) the odds, and line (10) becomes zero by combining terms with the same exponents and opposite signs (or opposite $sgn(\sigma)$, if you like).

3.2 Graph Intuition

In the general case, the difficulty is really line (10) - how do we do all the cancellations?

Rather than handle these $2^{\binom{n}{2}}$ terms by hand each time algebraically, through an isomorphism, we'll translate these factors $(x_j - x_i)$ as a set of $\binom{n}{2}$ *graphs*.

The $2^{\binom{n}{2}} = 2^{\binom{3}{2}} = 8$ terms expanded on lines (8) - (10) are clearly the 8 terms resulting from multiplication of the three factors $(x_2 - x_1)(x_2 - x_0)(x_1 - x_0)$. Each such algebraic term is isomorphic to one of the complete[3] directed[4] graphs (also known as "tournaments") in the set on 3 vertices, with the edge "pointing" from the selected term towards the omitted term in $(x_j - x_i)$.

For example, the term $x_2^2 x_1^1 x_0$, produced by multiplying the three left terms of the three factors of $(\mathbf{x_2} - x_1)(\mathbf{x_2} - x_0)(\mathbf{x_1} - x_0)$, corresponds to graph 1a.

[3] edges between every vertex

[4] an edge points toward one of its vertices

The term $x_2^1 x_1^1 x_0^1$, from multiplying the right, left, and right terms of the above product corresponds to 1b. And, in 1c, the inverted product sequence left, right, left, produces the inverted graph cycle and notably, the algebraic inverse $-x_2^1 x_1^1 x_0^1$ in the product expansion (sum).

This should give a flavor of the proof:
To find the expansion of P_I and show it equals S_I, we show each of the $2^{\binom{n}{2}}$ graphs produced when expanding P_I are isomorphic to a term in the sum like $-x_1^2 x_2^1 x_0^0$, where each exponent represents the "out degree" of a vertex x_i in the graph. If any of the exponents in the term corresponding to a graph are equal, we show the term drops out in the final cancellation, and through the same isomoprhism, we are left with S_I.

3.3 P-S Equivalence Lemma Proof layout

Here is a layout of the proof that $P_I = S_I$.

First, we prove a set of lemmas:

- (1) Lemma: There is an isomorphism between the set of terms in an expanded $P_{[0,n]} = \prod_{0 \le i < j \le n} (x_j - x_i)$ and the set of all possible tournaments of size $n + 1$.
- (2) Lemma: All tournaments are either acyclic or contain a 3-cycle.
- (3) Lemma: Acyclic tournaments correspond through the isomorphism with terms of the form $sgn(\sigma) x_{\sigma(n)}^n x_{\sigma(n-1)}^{n-1} ... x_{\sigma(0)}^0$ for some permutation σ on $[0, n]$.
- (4) Lemma: Cyclic tournaments with a 3-cycle can be paired 1:1 with an otherwise identical graph with that 3-cycle inverted.

Through these lemmas, we can start with a base case equality $P_{[0,2]} = S_{[0,2]}$ and show:

- (5) The isomorphism equates the set of edge configurations possible from adding an additional vertex x_n to an acyclic graph G of n vertices with the algebraic action of multiplying $\prod_{0 \le i \le n-1} (x_n - x_i)$ by $P_{[0,n-1]}$.
- (6) This isomorphism maps all possibilities of adding an additional vertex x_n to an acyclic graph G of n vertices to $S_{[0,n]}$. In particular, all terms corresponding to graphs with cycles cancel, and only those corresponding to acyclic tournaments remain in the sum.
- (7) Because $P_{[0,n]}$ and $S_{[0,n]}$ map to the same set of graphs *through the same isomorphism*, this shows the **P-S equivalency lemma**, and the inductive step of the Vandermonde Determinant Proof (section 2.1.2).

3.4 Lemma 1

There is an isomorphism between the set of terms in an expanded $P_{[0,n]} = \prod_{0\leq i<j\leq n}(x_j - x_i)$ *and the set of all possible tournaments of size* $n+1$*.*

Every possible complete directed graph $G = (E, V)$ of vertex size n consists exactly of edges (i, j)[5] with $i, j \in [v_0, v_n - 1], i < j$. If $(i \to j) \in E$, then consider $(\mathbf{x_i} - x_j)$ in the expansion of $P_{[0,n-1]}$; otherwise if $(j \to i) \in E$, then consider $(x_i - \mathbf{x_j})$ in the expansion of $P_{[0,n-1]}$. Conversely, if $(\mathbf{x_i} - x_j)$ is in a term of $P_{[0,n-1]}$, take $(i \to j)$ for an edge in the graph, otherwise $(j \to i)$. As in Figure 1, this isomorphism should be straightforward. [6]

Note: We call this bijective correspondence between an algebraic term in the expansion of $P_{[0,n]}$ *and a tournament on* $n+1$ *vertices "the isomorphism" throughout the paper. When we say a graph "pairs with an algebraic form" or a algebraic term "has a corresponding graph", it's meant to be through this bijection.*

3.5 Lemma 2

All directed complete graphs are either acyclic or contain a 3-cycle.

If the graph contains no cycles, or a cycle of length 3, we are done.

If a graph contains some cycle through vertices $(v_0 \to v_1 \to ... \to v_{m-1} \to v_0)$ of length $m > 3$, we can split it into two possible cycles: $A = (v_0 \to v_1 \to v_2 \leadsto v_0)$ and $B = (v_2 \to v_3 \to ... \to v_0 \leadsto v_2)$, with $\leadsto$ meaning "maybe goes to". Depending on the direction of edge (v_0, v_2), exactly one of A or B must be a cycle. If A is a cycle, we are done. Else use B and reapply recursively on this smaller cycle, eventually down to a cycle of length 3.

3.6 Lemma 3

Acyclic tournaments correspond through the isomorphism with terms of the form $sgn(\sigma)x^n_{\sigma(n)}x^{n-1}_{\sigma(n-1)}...x^0_{\sigma(0)}$ *for some permutation* σ *on* $[0, n]$*.*

Another way of saying "every acyclic tournament maps through some σ to$sgn(\sigma)x^n_{\sigma(n)}x^{n-1}_{\sigma(n-1)}...x^0_{\sigma(0)}$ is "No acyclic tournament has vertices of equal outdegree". For example, an acyclic tournament on a set of vertices indexed $[0, 3]$ necessarily looks like the following:

Every vertex here x_i has a unique outdegree $k \in [0, n-1]$, which, through the isomorphism, we see as the exponent of x_i in the graph's algebraic form. The above graph corresponds

[5] of some direction

[6] Or consider pulling out $Q = P_{[0,n]}/(x_i - x_j)$. Then $(x_i - x_j)Q = x_iQ - x_jQ$, two graph sets with different "selections" for edge (i, j)

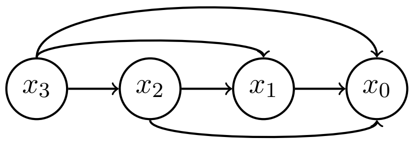


Figure 2: An acyclic tournament on 4 vertices

to the term $x_3^3 x_2^2 x_1^1 x_0^0$. As every edge here "points right", it's clear that there can be no cycle (or particularly, 3-cycle).

For the converse, consider the statement that "no acyclic tournament has a subgraph[7] with two or more vertices of equal degree". If this is true then certainly the graph has to have the above form of outdegrees. To see this:

- If the graph is of the form $sgn(\sigma)x^n_{\sigma(n)}x^{n-1}_{\sigma(n-1)}...x^0_{\sigma(0)}$ for some n and some σ, we are done.
- Suppose then it has two vertices with duplicate outdegrees, but has no cycles. Eliminate vertices, starting from x_n, then x_{n-1}, down to x_2 just until a subgraph is of the form with all unique outdegrees, which we'll call[8] $y_m...y_0$ with y_m of outdegree m and y_0 of outdegree 0. Call y_0, for now, y^-.
- Add the last removed vertex y^* and its edges back. y^* must have the same outdegree as some other vertex, otherwise we have a contradiction.
- Loop:: If $(y^- \to y^*)$ is in the graph, then necessarily there is a cycle $(y^- \to y^* \to$ some $y_j \to y^-)$, so we have a contradiction.
- Else $(y^* \to y^-)$ is in the graph, so the outdegree of y^* can't be 0. Remove y^-, reducing all vertices by outdegree 1, creating a new $y^- \neq y^*$ with minimum degree, and go back to (Loop).

From this, we can conclude that an acylic directed tournament has vertices of all unique degrees, and thus, up to vertex labeling (permutation), has the algebraic form $sgn(\sigma)x^n_{\sigma(n)}x^{n-1}_{\sigma(n-1)}...x^0_{\sigma(0)}$ for some σ.

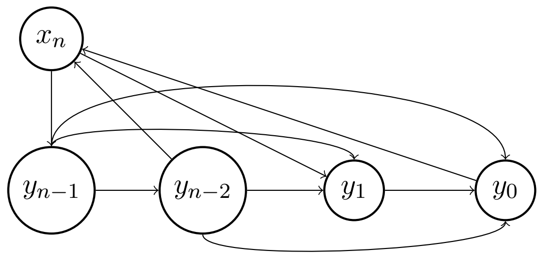
3.7 Lemma 4

Cyclic tournaments with a 3-cycle can be uniquely paired 1:1 with an otherwise identical graph with that 3-cycle inverted.

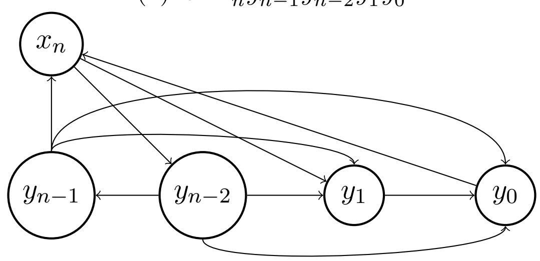
Suppose, for a graph G we fix an ordering of vertices like $\{x_n, x_{n-1}, x_{n-2}...x_0\}$. Suppose

[7] Meaning a copy of the original graph with some vertices and all their adjacent edges removed

[8] Note: the set of y are the same as the set of x, just with some other indexing



(a) G: $x_n^2 y_{n-1}^3 y_{n-2}^3 y_1^1 y_0^1$



(b) G: $-x_n^2 y_{n-1}^3 y_{n-2}^3 y_1^1 y_0^1$

Figure 3: Adding x_n can turn acyclic tournaments into cyclical ones

(x_a, x_b, x_c) is the 3-cycle with highest lexicographic order of the vertex set $(x_n > x_{n-1} > ...)$, with edges pointing either direction. Then, the graph G', with the same edges, except the direction of cycle (x_a, x_b, x_c) reversed:

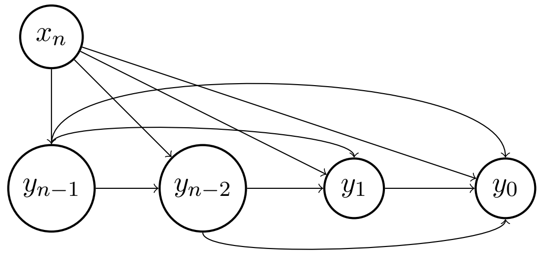
- (1) Can be mapped to uniquely to G, and vice versa.
- (2) Has an algebraic representation through the main bijection which is an inverse to that of G.

(1) is clear because each uniquely determines the other; the order of vertices is the same, thus the "first" cycle is the same, and the order need only be reversed. In Figure 3, the "first" cycle would be (x_n, y_{n-1}, y_{n-2}), even though the graph has many other cycles.
(2) Reversing $(\mathbf{x_1} - x_2)(\mathbf{x_2} - x_3)(\mathbf{x_3} - x_1) = x_1x_2x_3$ yields $(x_1 - \mathbf{x_2})(x_2 - \mathbf{x_3})(x_3 - \mathbf{x_1}) = -x_1x_2x_3$, for any choice of x_1, x_2, x_3. [9]

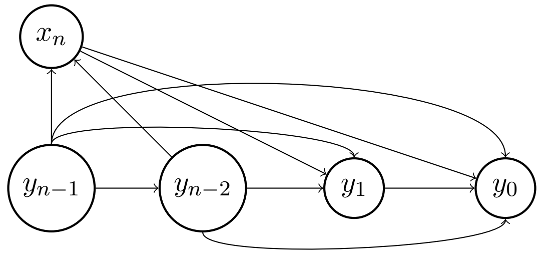
3.8 Step 5

The isomorphism equates the set of edge configurations in of adding an additional vertex x_n to an acyclic graph G of n vertices with the algebraic action of multiplying $\prod_{0\leq i\leq n-1}(x_n - x_i)$ by $P_{[0,n-1]}$.

[9] Note that in 3b, the graph without x_n is still a tournament by Lemma 3, as its algebraic representation has gone from $y_{n-1}^3 y_{n-2}^2 y_1^1 y_0^0$ to $-y_{n-2}^3 y_{n-1}^2 y_1^1 y_0^0$ with the reversal of edge (y_{n-1}, y_{n-2}).



(a) G: adding x_n to a tournament of 4 vertices $\simeq x^4 y_{n-1}^3 y_{n-2}^2 y_1^1 y_0^0$



(b) G: adding x_n to a tournament of 4 vertices: $\simeq y_{n-2}^4 y_{n-1}^3 x^2 y_1^1 y_0^0$

Figure 4: Adding x_n to an existing tournament on 4 vertices

By inductive assumption, suppose an acyclic tournament graph $G = (E, V)$ on a permutation of the vertices $y_{n-1}, y_{n-2}, ...y_0 \in V$ is constructed such that $(y_i \to y_j) \in E$ iff $i > j$.

Then, let's add a new vertex x_n to the tournament, which necessarily has connections to all of $\{y_{n-1}...y_0\}$ (Figure 4a).

Each of the 2^n terms (before any cancellation) resulting from $\prod_{0 \le i < n}(x_n - x_i)$ correspond 1:1 with one of the 2^n settings of the edges from x_n to every other x_i. These are coupled with the expressions $P_{[0,n]}$ and the graph on vertices $x_0, ...x_{n-1}$, respectively.

The above graph would correspond to the term $(\mathbf{x_n} - y_{n-1})(\mathbf{x_n} - y_{n-2})(\mathbf{x_n} - y_1)(\mathbf{x_n} - y_0)P_{[0,n-1]}$ in $S_{[0,n]}$.

3.9 Step 6

This bijection maps all possibilities of adding an additional vertex x_n to an acyclic graph G of n vertices to $S_{[0,n]}$.

Looking at Figures 3a and 3b, notice that, when adding x_n, if at any point there are edges $(x_n \to y_j), (y_i \to x_n), i < j$, then we necessarily have a cycle in the graph. By Lemma 4, these each are mapped 1:1 to another graph, which has an inverted "first cycle" and is an algebraic inverse. Thus, each pair of these contributes 0 to an expanded $P_{[0,n]}$.

So the only configurations of x_n's edges that do not create a cycle are those like $\{(y_{n-1} \to x_n), (y_{n-2} \to x_n)...(y_i \to x_n), (x_n \to y_{i-1}), ..., (x_n \to y_0)\}$, where all "ins" to x_n precede all "outs" from it, like Figures 4a and 4b. By Lemma 3, in the algebraic representation these correspond exactly to $sgn(\sigma)y_{n-1}^{n}y_{n-2}^{n-1}...y_{k}^{k+1}x_{n}^{k}y_{k-2}^{k-1}...y_{0}^{0}$.

Notice that the sum of these terms, over all permutations $\sigma \in Sym([0,n])$, is exactly $S_{[0,n]}$.

3.10 Step 7: Wrapping up

We assumed $P_{[0,n-1]} = S_{[0,n-1]}$.

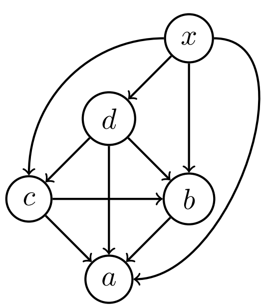
Expanding $P_{[0,n]}$ algebraically is difficult. Instead, we mapped it through the graph-algebraic isomorphism to a set of graphs. Separating cyclic and acyclic graphs, we saw, through the same isomorphism, that cyclic graphs cancel on the algebraic side, and acyclic graphs remained, leaving the sum of every permutation of
$sgn(\sigma)x_{\sigma(n)}^{n}x_{\sigma(n-1)}^{n-1}x_{\sigma(n-2)}^{n-2}...x_{\sigma(0)}^{0}, \sigma \in Sym([0,n])$.

This is $S_{[0,n]}$.

Having proved the **The P-S Equivalence Lemma**, we finish the inductive step of the Vandermonde Determinant Proof (chapter 2).

4 Example of adding x to a tournament of 4 vertices

Algebraically resolving $P_{[0,n]} = \prod_{0 \leq i,j, \leq n, j<i}(x_i - x_j)$ gets increasingly unwieldy to show, so we'll conclude with an example of adding all possibilities of x with directed edges to a single existing acyclic tournament[10] with order $d > c > b > a$, with representation $d^3c^2b^1a^0$.



The tournament represented by $g = x^4d^3c^2b^1a^0$

[10]When doing handwork like this, using a, b, c feels more natural than x_0, x_1, x_2...

The chart maps all 2^4 *terms* t in $p = (x-a)(x-b)(x-c)(x-d)$ multiplied by existing tournament representation $g = d^3c^2b^1a^0$, with the "product" $t \cdot g$ representing one of the 2^4 possible 5-vertex graphs.

These products either:

- Are isomorphic to an acyclic graph, so are of the form $sgn(\sigma)\sigma(x)^4\sigma(d)^3\sigma(c)^2\sigma(b)^1\sigma(a)^0$
- Are isomorphic to a graph with a cycle in $\{(x,d,c),(x,c,b),(x,b,a)\}$. If so, they have a uniquely matching pair t^* and graph g^* when flipping their first cycle.

t	$t \cdot g$	Matching factor t^* $\cdot$ Matching g^*	First cycle
x^4	$x^4d^3c^2b^1a^0$ none	none	
$-x^3a$	$-x^3d^3c^2b^1a^1$	$-x^3b \cdot -d^3c^2a^1b^0$	(x,b,a)
$-x^3b$	$-x^3d^3c^2b^2a^0$	$-x^3c \cdot -d^3b^2c^1a^0$	(x,c,b)
$-x^3c$	$-x^3d^3c^3b^1a^0$	$-x^3d \cdot -c^3d^2b^1a^0$	(x,d,c)
$-x^3d$	$-x^3d^4c^2b^1a^0$	none	none
x^2ba	$x^2d^3c^2b^2a^1$	$x^2ca \cdot -d^3b^2c^1a^0$	(x,c,b)
x^2ca	$x^2d^3c^3b^1a^1$	$x^2da \cdot -c^3d^2b^1a^0$	(x,d,c)
x^2da	$x^2d^4c^2b^1a^1$	$x^2db \cdot -d^3c^2a^1b^0$	(x,b,a)
x^2cb	$x^2d^3c^3b^2a^0$	$x^2db \cdot -c^3d^2b^1a^0$	(x,d,c)
x^2db	$x^2d^4c^2b^2a^0$	$x^2dc \cdot -d^3b^2c^1a^0$	(x,d,c)
x^2dc	$x^2d^4c^3b^1a^0$	none	none
$-xcba$	$-xd^3c^3b^2a^1$	$-xdba \cdot -c^3d^2b^1a^0$	(x,d,c)
$-xdba$	$-xd^4c^2b^2a^1$	$-xcba \cdot -d^3b^2c^1a^0$	$x,c,b)$
$-xdca$	$-xd^4c^3b^1a^1$	$-xdcb \cdot -d^3c^2a^1b^0$	(x,b,a)
$-xdcb$	$-xd^4c^3b^2a^0$	none	none
$dcba$	$d^4c^3b^2a^1$	none	none

This sum, $x^4d^3c^2b^1a^0 - x^3d^4c^2b^1a^0 + x^2d^4c^3b^1a^0 - xd^4c^3b^2a^0 + d^4c^3b^2a^1$, when added to the tables of all the other initial settings of g, produces $S_{[0,4]}$.

References

[1] Wikipedia: https://en.wikipedia.org/wiki/Vandermonde_matrix