Six not-so-easy Pieces

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July 31, 2025

Abstract

Wherein I attempt original proofs of non-original seminal graph theorems without reference to any other theorems or external input, for my own enjoyment.

Graph theory can sometimes admit systematic approaches, enabled by work like the *Spectral Theorem*, but by and large, each graph theory proof seems to be its own unique puzzle. This makes them great for the average swagman to attempt proofs of their theorems, many only discovered in the mid-20th century.

Though I solicited Claude Anthropic for interesting theorems to prove, this work is entirely my own, proven without reference to any other theorems. The lemmas may exist elsewhere or be entirely unnamed elsewhere. Thus, each proof may be inelegant, incomplete, or just plain wrong. At long last, I have run out of counterarguments for them.

Listed below are the statements of the six proofs, along with where I had the key insight. They are arranged in order of increasing difficulty (for me).

Menger's Theorem. Let $G = (V, E)$ be a finite undirected graph and A and B two disjoint subsets of vertices in V. Then the minimum number of vertices that need to be removed to disconnect A from B is equal to the maximum number of vertex-disjoint paths from A to B. *Moment of insight*: Child's swimming lesson.

König's Theorem In a bipartite graph, the size of the maximum matching equals the size of the minimum vertex cover (a set of vertices that together touch every edge). *Moment of insight*: Alone in the office.

Brooks's Theorem If a graph G is not a clique and not an odd cycle and has maximum degree d, it can be properly vertex-colored with d colors. *Moment of insight*: Bus.

Dirac's Theorem An n-vertex graph in which each vertex has degree at least $\frac{n}{2}$ must have a Hamiltonian cycle. *Moment of insight*: Bus.

Lovasz's Theorem: The complement of any perfect graph is perfect. *Moment of insight*: Child's swimming lesson.

Turan's Theorem: If a graph $G = (V, E)$ on n vertices satisfies $|E| > \frac{1}{2}n^2\left(1 - \frac{1}{k}\right)$, then G contains a $(k+1)$-clique. *Moment of insight*: Child's holiday concert.

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1 Menger's Theorem Original Proof

Theorem 1.1 (Menger's Theorem)**.** *Let $G = (V, E)$ be a finite undirected graph and A and B two disjoint subsets of vertices in V. Then the minimum number of vertices that need to be removed to disconnect A from B is equal to the maximum number of vertex-disjoint paths from A to B.*

Proof. On a graph G, define $n(G)$ as the number of vertex-disjoint paths between A and B, and the minimum cut as $m(G)$.

$m \geq n$**:** If we have $N = n(G)$ vertex-disjoint paths and we remove $M < n(G)$ of these vertices, A and B must still be connected through at least one path since no paths can share a vertex.

$n \geq m$**:** For any graph $G = (V, E)$, we prove by induction on induced subgraphs of size k (remove all but k vertices).

Base Case: $k = 0, 1$ are meaningless, with no paths available. So remove all but 2 vertices from G. If they are connected, $m(G) = n(G) = 1$. If not, $m(G) = n(G) = 0$. The condition holds.

Inductive Case: Say for any induced subgraph $G^* = (V^*, E^*)$ of G, where $|V^*| = k - 1, n(G^*) \geq m(G^*)$.

If adding a vertex v (and its edges incident to V^*) back into G^* does not increase the number of vertex-disjoint paths, then our condition holds, since removing v (adding it to the cut set) doesn't change $n(G)$ or $m(G)$.

If doing so *does* increase the number of vertex-disjoint paths, then it can do so only by one: Cutting v yields G^*, where the condition holds, and adding v may increase the cut set by one, but can only add one more vertex-disjoint path to G^* (informally, you can't use v more than once). □

Therefore $m(G) = n(G)$.

2 König's Theorem Original Proof

Theorem 2.1 (König's Theorem)**.** *In a bipartite graph, the size of the maximum matching equals the size of the minimum vertex cover (a set of vertices that together touch every edge).*

Proof. Assume we have vertices $\{a_i\}$ in one independent set and $\{b_i\}$ in the other independent set of the bipartite graph.

Relabel them so our maximum matching is $(a_1, b_1) \dots (a_m, b_m)$. This means there are sets of vertices $\{a_{m+i}\}$ $(i > 0)$ that only connect to $b_1 \dots b_m$ and vertices $\{b_{m+i}\}$ $(i > 0)$ that only connect to $a_1 \dots a_m$ (see Fig. 1).

If some $a_{m+j}, j > 0$ connects to b_i, $1 \leq i \leq m$, then no b_{m+k}, $k > 0$ can connect to a_i, since then we would have a bigger matching by removing (a_i, b_i) and adding (a_{m+j}, b_i) and (a_i, b_{m+k}), a contradiction of our assumptions.

Therefore, for every vertex in the a-component outside the matching, add to the cover any adjacent b_i in the matching (and vice versa for every b vertex outside the matching). And there can be no vertices in the b-component outside the matching connecting to a_i, so only one of a_i, b_i will be selected. If a_i and b_i have no adjacencies outside the matching, or have not been selected after all the non-matched vertices are covered, then select either one for the cover. Therefore, we will have exactly one vertex in the cover for every edge in the matching.

This vertex cover covers all vertices outside the original matching and clearly covers all the vertices $a_1 \dots a_m, b_1 \dots b_m$ too. It is the minimum as any vertex set smaller than size m cannot cover the vertices in the matching.

□

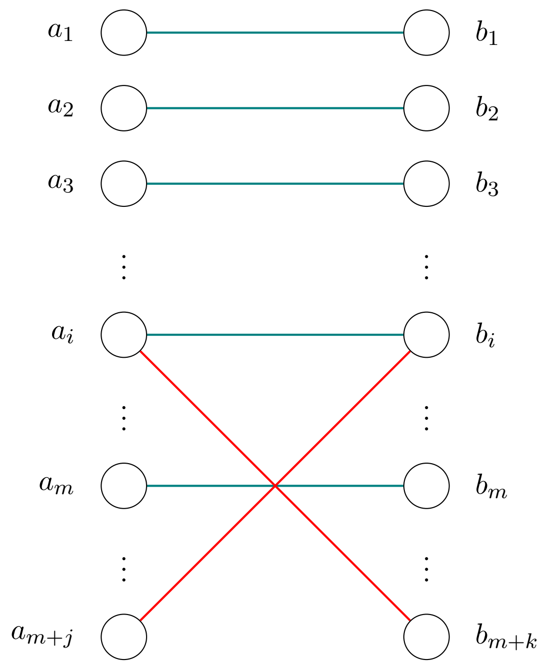

Figure 1: Green is the maximum matching. Red edges as such cannot exist.

3 Brooks's Theorem Original Proof

Theorem 3.1 (Brooks)**.** *If a graph G is not a clique and not an odd cycle and has maximum degree d, it can be properly vertex-colored with d colors.*

First, some lemmas. Call graphs which are not themselves odd cycles or complete graphs *allowed graphs.*

Lemma 3.2 (Augmented odd cycle colorability)**.** *If an allowed graph G consists of an odd cycle C_n plus an additional vertex v, it is colorable with maximum degree $= d$ colors.*

Proof: $G - v = C_n$ is colorable in three colors, an alternating chain of $\frac{n}{2}$ red, blue pairs, plus a final green vertex. Since G is an allowed graph, v is connected to C_n, and therefore some vertex in C_n has degree 3.

If $deg(v) \geq 4$, color v yellow, and $\chi(G) \leq 4 \leq d$. If $deg(v) \leq 2$, a color is clearly available for v, so $\chi(G) = 3 = d$. If $deg(v) = 3$, we consider first $n = 3$. This is disallowed since G would be a complete graph. Then since $n \geq 4$, there is a vertex unconnected to v. Rotate the colors of C_n so this one is green and color v green. Then $\chi(G) = 3 = d$.

Lemma 3.3 (Augmented complete graph colorability)**.** *If an allowed graph G consists of a complete graph K_n plus an additional vertex v, it is colorable with maximum degree $d = n$*

colors.

Proof: K_n *is clearly colorable in* n *colors. Since* $G = (V, E)$ *is an allowed graph,* $0 < deg(v) < n$*, and* v *is connected to* K_n *in* G*. This means some vertex in* K_n *must have degree* n*. Additionally, since* G *is an allowed graph, there must be some vertex* $w, (v, w) \notin E$ *with degree* $= n - 1$*. Color* v *the same as* w*. The graph is colorable in* n *colors, with max degree* n*.*

Proof of Brooks's theorem:

Note: We assume this is a connected graph; if it's a disconnected graph, then the maximum degree of any component will be less than or equal to that of the graph, and we can color each component separately by the procedure below. We also ignore the degenerate graph of a set of isolated vertices.

We will take any allowed G and proceed by induction on induced subgraphs $G^* = (V^*, E^*)$ of G.

Base Case: $|G^*| = 2$: If connected, the graph is a single pair, a clique of size 2, colorable with two colors.

Inductive Case: Assume for all induced subgraphs of G^* of $G = (V, E)$, the proposition holds and G is not an odd cycle or a complete graph and has max degree $d(G)$.

Remove a vertex v of lowest degree in G. By inductive hypothesis the induced graph $G^* = G - v$, with max degree $d(G^*) \leq d(G)$, is colorable in $d := d(G^*)$ colors.

Also, we assume $d \geq 3$, since otherwise we are done:

- If $d = 1$, we have a set of pairs. If $deg(v) = 1$, G is 2-colorable. If $deg(v) = 2$, and it is connected to two vertices in a pair, we have a disallowed component of graph G. If $deg(v) = 2$ and it is connected to two different pairs, that path is two-colorable. If $deg(v) \geq 3$, color the pairs red-blue and v green.
- if $d = 2$, and does not include an odd cycle component (handled below), then including a connected v brings max degree of G to at least 3. Color the even cycles red-blue and v green.

Let's also treat easy cases of the induction:

- If any induced graph component is a cycle, then the cycle plus v is colorable in 3 colors by the Augmented Odd Cycle lemma or the argument above for $d = 2$ (even cycle). Since $d(G) \geq 3$, this component is covered.
- If any induced graph component is a complete graph, then adding v makes it colorable within bounds by the Augmented Complete Graph lemma. This component is covered.

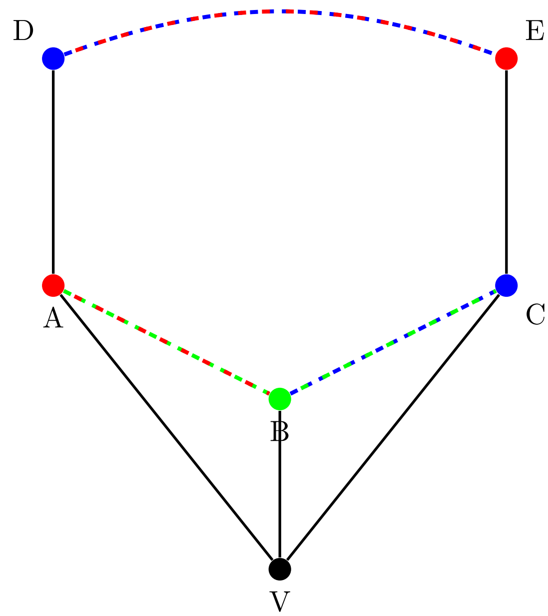

Figure 2: Disrupt any alternating path and colorability follows

- If the maximum degree $d(G^*)$ greater than $deg(v)$, color the induced graph with $d(G^*)$ colors, and there will be at least one left for v.

Because of the last case, we assume every vertex in $G*$ has degree at most $deg(v)+1$.

We will illustrate this with $d(G^*) = 3$ (thus $d(G) \leq 4$) and then argue this generalizes to $d(G^*) > 3$.

Include v, uncolored, back into the graph G^* as shown:

If $d(G^*) = 4$, we ignore the presumed 'yellow' vertex connected to v for now (see argument for 'only three colors matter' at the end).

Our goal is to change the number of colors among $\{A, B, C\}$ to less than three. If that is the case,then v has a free one to select and we're done.

More assumptions:

- If two of v's neighbors share a color already, then we're done. Therefore, assume v's neighbors A, B, and C are colored red, green, and blue, respectively.
- (*) We show paths between A and C (ignoring D and E for the moment), A and B, and B and C, possibly just a single edge. They cannot all be single edges, however. If so, $G = \{A, B, C, V\}$ was a complete subgraph K_4. If disconnected from the rest

of the graph, we have a disallowed condition. If one of these vertices shares an edge outside of this set with some vertex w, then the max degree of G^* is 4 and v can be colored yellow. So at least one of the three paths has more than one edge (and since alternating colors, must have at least three edges). Here, without loss of generality, we make it the red-blue path from A to C. (Note: if $deg(v) = 4$, the same logic applies: there must be a selection of A, B, and C such that two of them are not directly connected.)

- If there is not an alternating red-blue path from A to C, then switch red and blue along that path and and transitive neighbors. The graph will still be valid, with A and C both blue. Color v red.
- The same logic applies for red-green paths between A and B, and blue-green paths between B and C.
- So assume there are these three alternating color paths as shown in Fig. 2.

If we can flip exactly one endpoint of any of these three paths $(A, B), (B, C), (A, C)$, we can color v freely.

We introduce another other lemma to help us do our flipping:

Lemma 3.4 (Splitting Lemma)**.** *If an alternating color path encounters a vertex where two successive (yet unflipped) vertices share a color, then then transitive chain starting from the beginning to this point can be flipped.*

Proof: Refer to figures. If in, say a red-blue chain (Fig. 3), w is transformed to red, x has been transformed to blue, and y, z are both green, the chain must stop. If $deg(x) = 3$, it can stay blue while not violating color constraints. If x is adjacent to an additional blue vertex, color x yellow since $deg(x) = 4$. if y and z are both blue, there is at least one other color to change x to (Fig. 4) If there is an additional vertex connected to x and it is 'yellow' (fourth color), the above arguments don't change. If it is not 'yellow', color x yellow.

In particular, this means that if chains of two colors SPLIT into child chains, the original chain can be flipped up to the point of intersection.

Corollary 3.5 (No crossing Lemma)**.** *If two alternating chains cross, they are both flippable at the point of intersection.*

Similarly to the splitting lemma, this means there are four vertices adjacent to the point of intersection (Fig. 5), and therefore at least one color unrepresented in the neighbor set, and the intersection can be colored with one of these.

Coloring the figure

Consider the first figure again, and consider a blue-green chain from vertex D.

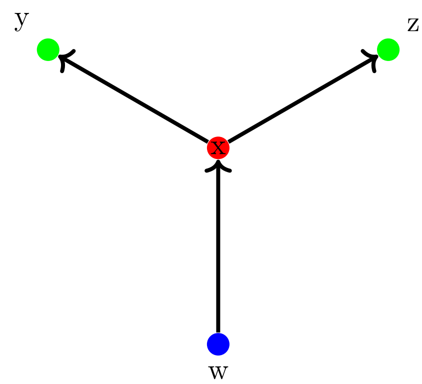

Figure 3: Flippable if $w - x$ follows a red-blue chain.

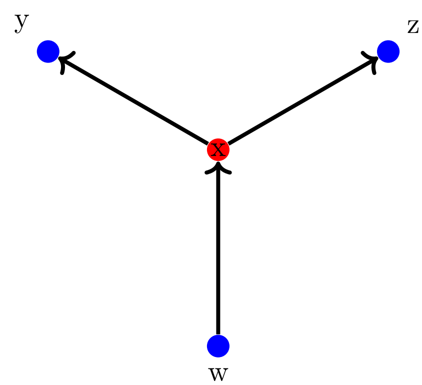

Figure 4: Also flippable if $w - x$ follows a red-blue chain.

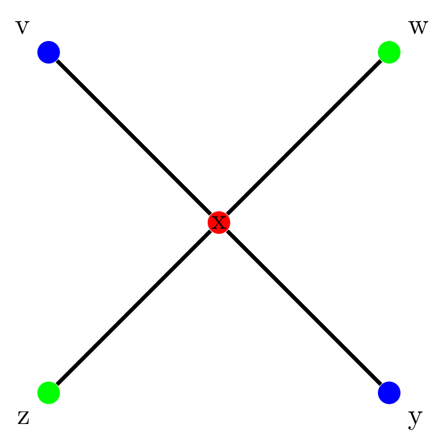

Figure 5: If two flipping chains cross, they can stop at any intersection

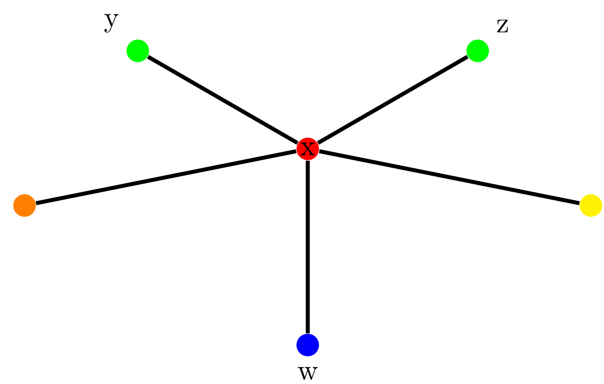

Figure 6: Additional colors can be ignored. Any duplicates allow flipping as usual.

- There must be a blue-green chain (of some positive length) from D, otherwise D has three red neighbors, and the splitting lemma applies when trying to draw a red-blue chain from A to C.
- That blue-green chain cannot intersect the chain from B to C, otherwise the splitting lemma applies when trying to draw a blue-green path from B to C.
- If the blue-green chain does not intersect the red-green chain from A to B (or splits on the way, meaning D can be made green), A has two green neighbors and can be colored blue (or, if it has three neighbors, color the remaining color). Color v red.
- If the blue green chain DOES intersect the red-green chain from A to B at some formerly green vertex w, and if w is not adjacent to A, A still has two green neighbors and can thus be flipped.
- If in the last case, w is adjacent to A, then A has a blue neighbbor and a green neighbor. However, C now either connects to A (through E) through a split path, $deg(A) = 4$ and the other vertex is yellow (no change in argument), $deg(A) = 4$ and A connects to two of the same color (flipping argument), or not at all. So C can be flipped to red (or yellow in the third case), and V to blue.

Lemma 3.6 (Only three colors matter)**.** ***Generalizing to*** $d > 3$ *When flipping a path starting from A, B, or C, if the number of colors available is less than d, we can always end the chain there (Fig. 6). This means, no matter how many colors beyond three we have, if a chain can "split", or if every vertex doesn't have* $d - 1$ *colors adjacent to it (excluding its own color), then we can stop our path flipping at that spot and convert the color of origin (in our case, A, B, or C).*

Therefore, even if $d(G^*) = 3 + j, j > 0$, we can ignore all of the j colors outside of these three when constructing these paths. The argument for at least one cycle of degree greater than three (marked (*) above) holds as well, since complete graphs of degree d follow the same argument.

4 Dirac's Theorem Original Proof

Theorem 4.1 (Dirac)**.** *An n-vertex graph in which each vertex has degree at least $\frac{n}{2}$ must have a Hamiltonian cycle.*

First, some lemmas:

Lemma 4.2 (Minimum cycle length)**.** *If a graph has vertices all of degree $d \geq 2$, there exists a cycle within it of length at least $d + 1$.*

Proof: A greedy algorithm suffices here. Select any starting vertex v. Define the "seen" vertex list Las $\{v\}$ and head vertex h as v. $N(h)$ is a function defined as the neighbor set of some vertex h.

1. If $N(h) \subseteq L$, pick the earliest vertex in the list e. Then the cycle is the slice of L between e and h, with h connecting to e at the end. Because h has degree $\geq d$, e must be at least d vertices back in the list, making the cycle length at least $d + 1$.

2. Else append h to L and set h to one of its neighbors not in L.

Suppose there is a graph of G of even vertex count $n > 4$.

There must be a cycle of size s at least $\frac{n}{2} + 1$ by the Minimum cycle length lemma. Call this component C and the complement $D := G - C$.

Lemma 4.3 (Crowded neighbors principle)**.** *: If there are n vertices in a cycle, selecting $\lfloor \frac{n}{2} \rfloor + 1$ of them must yield at least two neighboring vertices.*

Proof: Selected vertices cannot outnumber unselected vertices since every selected vertex must be followed by an unselected one in a cycle without neighbors being selected.

For Dirac's theorem, we first dispatch with small vertex counts:

- This is meaningless for $n \in 1, 2$. No cycles are possible.
- This is clear for $n = 3$: a triangle.
- For $n = 4$, connected vertices a and b must be connected to c and d respectively (blue edges in figure), or both to some c (red edges). See Fig. 7.
  - If the first (blue) case, if c is connected to d, we have a 4-cycle. If c is connected to b, then d must be connected to a or c, both yielding cycles.
  - if the second (red) case, d is connected to two of $\{a, b, c\}$, creating a 4-cycle.

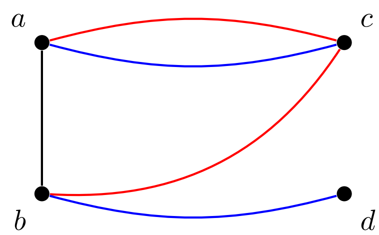

Figure 7: $n = 4$ necessitates a Hamiltonian cycle when all degrees ≥ 2

Lemma 4.4 (Dirac's Theorem,vertex count $n \geq 4, n$ even)**.** *: Dirac's theorem holds for $G = (V, E)$, where vertex count $n := |V| \geq 4, n$ even.*

Proof:

First, in such a graph there must be a cycle of size s at least $\frac{n}{2} + 1$ by the Minimum Cycle Length theorem. Call this set of cycle vertices C and the remaining vertices D. Assume D is non-empty (otherwise s is a Hamiltonian cycle). Therefore, $|s| = |C| \leq n - 1$.

Also, two definitions:

- For vertex $d \in D$, define the out-degree $out(d)$ to be the number of vertices d is connected to in C.
- For vertex $d \in D$, define the in-degree $in(d)$ to be the number of vertices d is connected to in D.

We seek to prove that if the degree of every vertex is $\geq \frac{n}{2}$ and D is non-empty, a cycle larger than s can be formed. With that, we set the larger cycle to C and remove those vertices from D. We then repeat until D is empty and C is a Hamiltonian cycle.

Let k be the minimum in-degree of the vertices in D.

- If $k = 0$, then some vertex $d \in D$ has out-degree $\geq \frac{n}{2}$. Since $|s| \leq n - 1 < 2 \cdot \frac{n}{2}$, by the Crowded Neighbors Principle, d must adjoin two adjacent vertices $a, b \in s$. A longer cycle than s is $(a, \ldots, b, v, a)$, with a, b connected "the long way 'round".
- If $k = 1$, then $|s| \leq n - 2$, and some vertex $d \in D$ has out-degree $\geq \frac{n}{2} - 1$, If $|s| < n - 2 = 2(\frac{n}{2} - 1)$, two vertices in s are adjacent to d by Crowded Neighbors. If $|s| = 2(\frac{n}{2} - 1)$, any two of d's adjacencies in s, say c_1, c_2 must all be at least two edges apart (not adjacent) in the cycle for no greater cycle to form; otherwise the path $(c_1, \ldots, c_2, d, c_1)$ is longer than $(c_1, \ldots, c_2, c_1)$, again, going "the long way 'round". This means d is connected to exactly "every other vertex" in s.

  Then d's neighbor $d_1 \in D$ must have at least one adjacency in s, call it c_1. However, c_1 cannot be more than two vertices apart from some d-connection $c \in s$. Adding

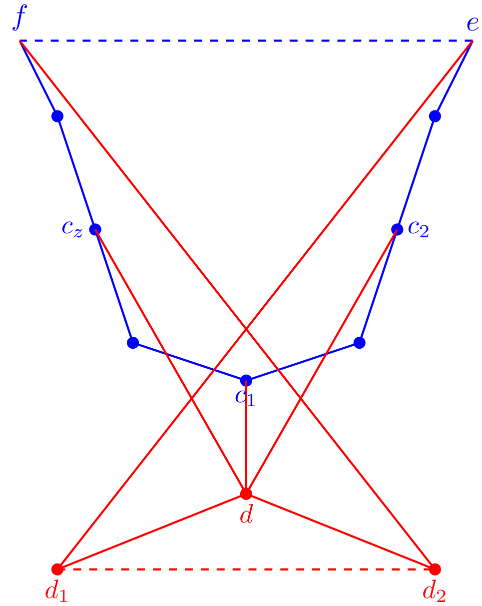

Figure 8: There is not enough room for d_1, d_2 to not create a bigger cycle

the path (c, d, d_1, c_1) to s and removing the path between c and c_1 (max length two) increases the cycle length of s.

- If $k \geq 2$, there is a cycle of length at least $k+1$ in D by the minimum cycle lemma.

  There is some vertex $d \in D$ with degree k in a cycle of length at least $k+1$ (to see this, apply the minimum cycle length theorem algorithm starting with d). This means d has $\frac{n}{2} - k$ neighbors in C, which must span a path of at least $2(\frac{n}{2} - k) - 1$ vertices if they are each at least two apart (otherwise, adding the 'jump out to d' and removing the original cycle edge increases cycle length). (See Fig. 8).

  But d also has two neighbors in D: d_1 and d_2, in a cycle of length at least $k+1$, so d_1 and d_2 have a path of $k-1$ vertices between them, inclusive. d_1 has at least one neighbor in C, e which must be $k+1$ away from any of d's neighbors d_i (otherwise, if d_i connects to e, and e is less than $k+1$ vertices away from some c_i along s, then $(e, d_1, \ldots d_2, d, c_i)$ is a cycle of length $k+2$ or greater). This is the same for d_2 with neighbor f. This means that C must contain $2(fracn2 - k) - 1 + 2(k+1) = n+1$ vertices, which it does not.

Case where n is odd, $n > 4$: If n is odd, then remove any vertex a. This reduces the degree requirement for all vertices by one (e.g. degree greater than 7/2 means 4, and degree greater than 6/2 means three). There must be a Hamiltomian cycle among the $n-1$

vertices by the even case.. But a is must be connected to two adjacent vertices b and c as its requirements mean it is connected to *more* than half of the remaining vertices. By the Crowded Neighbor principle, there are two adjacent vertices in s, b and c, connected to a. Remove the edge (b, c) from s and insert the path (b, a, c) and we have a Hamiltonian cycle.

5 Lovasz's Theorem Originial Proof

Definition 5.1 (Perfect Graph)**.** *A weakly perfect graph G has a chromatic number $\chi(G)$ (the number of colors required for a vertex coloring) equal to its clique number $\omega(G)$ (the size of its largest fully-connected subgraph or* clique*). A refinement of this, a strongly perfect graph G, has this property for every induced subgraph $G^* \subset G$.*

Theorem 5.2 (Lovasz)**.** *The complement of any perfect graph is perfect.*

Let's start with a seemingly smaller theorem: A cobipartite graph (complement of a bipartite graph) is perfect. Since a bipartite graph is clearly perfect (chromatic number of two, clique number of two), showing this proves Lovasz's theorem for a subset of graphs.

In a cobipartite graph, there are two cliques, K_m and K_n, with some set of connections between them.

Denote the maximum clique $C_m \cup C_n$, where $C_m \subseteq K_m$, $C_n \subseteq K_n$, and call M the remaining vertices in K_m. These are the vertices we need to color in $|C_n|$ colors to show$K_m \cup K_n$ is a weakly perfect graph.

Define the fanout $F(S)$ of a subset of $S \subseteq M$ as the set of vertices in M that are NOT connected to C_n. (This is an odd name, but it defines the set of colors M can take).

Lemma 5.3 (Expanding fanout lemma)**.** *$|F(S)| \geq |S|$ for all $S \subseteq M$.*

Proof: Suppose that $|F(S)| < |S|$ for some $S \subseteq M$. This means that S is connected to all of C_m (since $M \subset K_m$), and everything in C_n except some set $F(S)$. However, the clique $S \cup C_m \cap (C_n - F(S))$ is larger than $C_m \cap C_n$ since $|F(S)| < |S|$, which contradicts the maximality of $C_m \cap C_n$.

This also implies that $|F(M)| \geq |M|$, so $|C_n| \geq |M|$.

Lemma 5.4 (Fanout colorability lemma)**.** *For every such $S \subseteq M$ with fanout $F(S)$ where $|F(S)| \geq |S|$, S can be colored with the colors of $F(S)$.*

We proceed by induction.

Base case: If $|S| = 1$, then there is at least one vertex in C_n unconnected to S, and we color it trivially.

Inductive step: The inductive step will be by contradiction. Suppose this is true for all $|S| < k$ but NOT true for $|S| = k$. First, if there is a subset of T of S where $|T| = |F(T)|$, then $|F(S-T)| \geq |S-T|$, and we have two smaller sets we can color separately by inductive hypothesis. Thus, we assume $|F(T)| > |T|$ for all $T \subset S$.

If T and $S - T$ both have fanouts larger than they, then they have to intersect in some vertex $X \in F(S)$. Color this vertex in T the color of the corresponding vertex in C_n (remember, they are the ones NOT connected to C_n) and remove it from T and $S - T$. These reduced sets are colorable by inductive hypothesis.

Therefore, M can be colored with the colors of C_n, and by symmetry, N can be colored with the colors of C_m.

Lovasz: Say a graph is perfect and colorable in k colors. This is actually a k-partite graph, where each partition corresponds to one of the k colors. Its complement has cliques $K_1 ... K_k$ with some maximal clique among them.

The logic of the cobipartite graph applies completely; when discussing some M excluded from the maximal clique, it did not matter how C_n was distributed among other partitions. In a bipartite graph, they were all in a second partition, but in a k-partite graph, they could just as easily be spread among $k - 1$ partitions. So each excluded subset M_i of a partition can be colored with the colors of its fanout (among all $K_j, j \neq i$), no matter where they are grouped, and we have a perfect graph. We can say that a weakly perfect graph has a weakly perfect complement and by the Expanding Fanout Lemma, which will hold on every induced subgraph, a strongly perfect graph has a strongly perfect complement.

6 Turan's Theorem Original Proof

Theorem 6.1 (Turan)**.** *If a graph $G = (V, E)$ on n vertices satisfies $|E| > \frac{1}{2} n^2 \left(1 - \frac{1}{k}\right)$, then G contains a $(k + 1)$-clique.*

Define:

$$f(n, k) = \frac{1}{2} n^2 \left(1 - \frac{1}{k}\right).$$

Contrapositive. If G has no $(k + 1)$-clique, then

$$|E| \leq f(n, k).$$

Proof Plan: If we can transform any graph $G = (V, E)$ without a $(k + 1)$-clique into a graph $G^* = (V, E^*)$ where G^* has no $(k + 1)$-clique and $|E| = |E^*| \leq f(n, k)$, the contrapositive of Turan's theorem follows directly.

We consider such a set of graphs next: k-partite graphs. An example of these is bipartite graphs ($k = 2$).

Lemma 6.2 (k-partite bound)**.** *A k-partite graph on n vertices contains no $(k+1)$-clique and satisfies*

$$|E| \leq f(n, k).$$

(The proof of the k-partite bound lemma is in a later section.)

Lemma 6.3 (Transformation lemma)**.** *Let G be a graph on n vertices with no $(k+1)$-clique. Then there exists a k-partite graph on the same vertex set with the same number of edges, in particular:*

$$|E| \leq \frac{1}{2}n^2\Big(1 - \frac{1}{k}\Big),$$

Additionally, G possesses a vertex of degree at most $n\big(1 - \frac{1}{k}\big)$.

Base case

For $n \leq k+1$ the claim is immediate: the complete graph K_{k+1} contains $\frac{1}{2}(k+1)k$ edges. If we have less than or equal to $\frac{1}{2}(k+1)k - 1$ edges, we cannot form a $k+1$-clique. We show that this edge count is less than the bound.

$$-\frac{1}{2}k^2 - \frac{k}{2} + \frac{1}{2} \leq 0 \tag{1}$$

$$\frac{1}{2}k^3 + \frac{1}{2}k^2 - k \leq \frac{1}{2}k^3 + k^2 - \frac{1}{2}k - \frac{1}{2} \tag{2}$$

$$\frac{1}{2}k^3 + \frac{1}{2}k^2 - k \leq \frac{1}{2}(k^2 + 2k + 1)(k - 1) \tag{3}$$

$$\frac{1}{2}(k+1)k - 1 \leq \frac{1}{2}(k+1)^2\frac{k-1}{k} \tag{4}$$

(1) is clear since $k > 1$.

Then, any graph avoiding a $(k+1)$-clique already satisfies the bound and is itself k-partite (some parts may be empty).

Because $|E| \leq \frac{1}{2}n^2(1 - \frac{1}{k})$, the average degree is at most $n(1 - \frac{1}{k})$; hence some vertex has degree at most $\lfloor(n(1 - \frac{1}{k})\rfloor$.

Inductive step

Assume $n > k+1$ and that the lemma holds for all graph of vertex count n or fewer. Let G be a graph on $n+1$ vertices with no $(k+1)$-clique.

Every induced subgraph of size n also avoids $(k+1)$-cliques and so meets the edge bound $f(n,k)$.

Goal: find a vertex v of degree

$$\deg(v) \leq D := n - \left\lfloor \frac{n}{k} \right\rfloor.$$

If so, we apply the induction hypothesis to $G - v$ to obtain a k-partite graph, then assign v to a smallest partition (necessarily of size $\leq \lfloor \frac{n}{k} \rfloor$), rewiring its at most D edges across the remaining partitions. The resulting graph is k-partite with the same edge count as G.

Lemma 6.4 (Degree bound lemma)**.** *The above-mentioned bound $\lfloor n(1-\frac{1}{k})\rfloor$ is equal to the bound target bound D if $k \mid n$ and equal to $D-1$ if $k \nmid n$.*

If $k \mid n$ then

$$n - \left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor n\Big(1 - \frac{1}{k}\Big) \right\rfloor,$$

the floor operation becomes the identity and equality follows.

If $n = qk + r$, $\lfloor n(1-\frac{1}{k})\rfloor = \lfloor (qk+r) - q - \frac{r}{k} \rfloor$, and $n - \lfloor \frac{n}{k} \rfloor = qk + r - \lfloor (q + \frac{r}{k}) \rfloor$.

Taking out the integer $qk+r$, $\lfloor -q - \frac{r}{k} \rfloor < -\lfloor q + r/k \rfloor$, and they lie on either side of integer $-q$, so $\lfloor n(1-\frac{1}{k})\rfloor = n - \lfloor \frac{n}{k} \rfloor - 1$.

Existence of a small degree vertex

- If $k \nmid n$, choose any subset of n vertices of G, excluding some vertex v,

  By induction the subset contains a vertex w of degree $\leq n(1-\frac{1}{k})$, which is $\leq D-1$ by the Degree Bound Lemma, so the whole graph has such a vertex. Even if $(v,w) \in E$, w satisfies the degree bound.

- Suppose now $k \mid n$ (no more floors!) and, toward contradiction, that every vertex satisfies $\deg(v) \geq D+1$.

  Count edges in all $\binom{n+1}{n} = n+1$ induced n-vertex subgraphs. By induction each has at most

  $$f(n,k) = \frac{1}{2} n^2 \Big(1 - \frac{1}{k}\Big)$$

  edges. Summing and dividing by the multiplicity $n-1$ with which each edge is counted (two subgraphs will contain its endpoints), we obtain an upper bound

  $$T_{\text{upper}} = \frac{n+1}{n-1} f(n,k).$$

On the other hand, our degree assumption forces at least

$$T_{\text{lower}} = \frac{n+1}{2}\big(n - \tfrac{n}{k} + 1\big)$$

edges in total. Compute

$$T_{\text{lower}} - T_{\text{upper}} = \frac{n+1}{2}\,\frac{n-k}{k(n-1)} > 0$$

because $n > k$. This contradiction establishes that a vertex of degree $\leq D$ exists.

Completion of the inductive step

Removing the low-degree vertex and applying the rewiring argument yields a k-partite graph on $n+1$ vertices with the same edge count as G. Hence every $(n+1)$-vertex graph without a $(k+1)$-clique satisfies

$$|E| \leq \frac{1}{2}n^2\Big(1 - \frac{1}{k}\Big).$$

Corollary (Turan). If $|E| > \frac{1}{2}n^2(1 - \frac{1}{k})$ then G must contain a $(k+1)$-clique.

Proof of k-partite lemma

A k-partite graph has k partitions of vertices, within which all are disconnected. The vertices from differing partitions may connect. Clearly, a $k+1$-clique cannot exist among them, as vertices within each partition are disconnected, and there are only k partitions.

A perfectly balanced graph has partitions all of equal size.

Proposition 6.5. *The edge-count formula of a perfectly balanced k-partite graph is*

$$E_p(n,k) = \frac{n^2}{2}\Big(1 - \frac{1}{k}\Big).$$

Each of n vertices connects to $n\frac{k-1}{k}$ vertices in other partitions. We divide by two to get the number of edges.

Proposition 6.6. *The edge-count formula of a perfectly balanced graph of vertex total n is the upper bound to any partitioning of vertices among k partitions.*

We are looking at maximizing $f(\vec{a}) = \frac{1}{2}\sum_{i \neq j; i,j<k}^{k} a_i a_j$, where a_i is the vertex count of partition i.

We can write this as

$$f(\vec{a}) = \frac{1}{2} \sum_{i \neq j; i,j < k} a_i a_j = (\sum_{i}^{k} a_i)^2 - \frac{1}{2} \sum_{i}^{k} {a_i}^2 = n^2 - \frac{1}{2} \sum_{i}^{k} {a_i}^2 \tag{5}$$

So we are looking at minimizing the final term to maximize $f(\vec{a})$

Consider the definition of statistical variance:

$$Var(a_1, a_2 ... a_k) = \sum_{i=1}^{k} (a_i - \bar{a})^2 = \sum_{i=1}^{k} (a_i^2 - 2\bar{a}a_i + \bar{a}^2) \tag{6}$$

So $Var(a_1, a_2 ... a_k) = \frac{1}{k}[\sum_{i=1}^{k}(a_i^2) - 2\bar{a}(\sum_{i=1}^{k} a_i) + k\bar{a}^2]$

Note that $2\bar{a}(\sum_{i=1}^{k} a_i) = 2k\bar{a}^2$.

Thus, ignoring constant terms, minimizing $Var(a_1, a_2 ... a_k)$ is the same as minimizing $\sum_{i=1}^{k} {a_i}^2$. But the variance reaches its minimum at zero when all values a_i are equal. This is therefore bounded by the edge count of the equal partition $E_p(n, k) = \frac{n^2}{2}\left(1 - \frac{1}{k}\right)$.

Conclusion

Therefore, any k-partite graph has less than or equal to $\frac{n^2}{2}\left(1 - \frac{1}{k}\right)$ edges.