Proof of the Friendship Theorem

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Abstract

A proof of the "Friendship Theorem", the problem proposed by Claude, solved by yours sincerely after many misstarts.

Theorem 1 (The Friendship Theorem)**.** *If everyone at a party has exactly one friend in common, then there is a person who is friends with everyone else.*

Proof. Naturally, we model this as a graph. Here is such a graph, where $v_2 ... v_5$ all share v_1 as a friend, and between v_1 and any other partygoer, one friend exists in common.

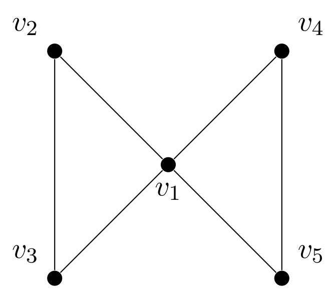

Figure 1: Bowtie graph

Another way to think about this: there is exactly one 2-path between any two different vertices.

First, we should render the party as an adjacency matrix M of size n by n, which, by definition, will:

- have values only in $\{0, 1\}$ (0 for nonadjacency, 1 for adjacency),
- be symmetric,
- and have zeros on the diagonal (no self-loops).

Here is the adjacency matrix for Fig. 1:

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Notice that outside of the first column and first row, there is exactly one 1 in each row and column.

Since squaring an adjacency matrix M will produce the number of 2-paths between vertices i and j at $M_{i,j}$ and $M_{j,i}$, M^2 will be of the form $J - I + D$, where J is the all-ones matrix (every vertex has exactly one 2-path to every other), and D is a diagonal matrix (every vertex v_i has $d(v_i)$ 2-paths to itself).

Here, for example, is M^2 for Fig. 1:

$$M^2 = \begin{bmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

We proceed by contradiction. Assume the vertex with the highest number of neighbors has degree less than $n-1$. Without loss of generality, reassign this as v_1, and all its neighbors as $v_2 \dots v_m, m < n$, so the first row of M reads as $M_{1,*} = [0, 1, 1..., 0...0]$ (and, by the symmetric nature of the matrix, the first column as its transpose).

$$M = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1_m & 0 & \cdots & 0_n \\ 1 & 0 & & & & & & \\ 1 & & & & & & & \\ \vdots & & & \ddots & & & & \\ 1_m & & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & \ddots & \\ 0_n & & & & & & & \end{bmatrix}$$

We know the first row of M^2 must be $[x, 1, 1, 1...1_n]$ for some x, and the first column its transpose, by the symmetric property.

This means that every row $\vec{r_i} = M_{i,*}, i > 1$ must have exactly one 1 among $\{r_2 ... r_m\}$ with the rest zeroes, to produce $\vec{r_i} \cdot M_{*,1} = 1$. This, by symmetry, means every column of M, outside the first row, must also have exactly one 1 in the entries $M_{1,i} .... M_{m,i}$.

However, since $m < n - 1$, and every such 1 entry guarantees the other entries from index 2 to m in its column (and its row) are zero, this means we have to fit $n - 1$ row- and column-isolated ones into a square submatrix of size $m < n - 1$, which is impossible.

Therefore, we cannot have $m < n - 1$, so $m = n - 1$ is the only possibility. A bowtie graph for any odd n will suffice. In every such party, a member has exactly one friend in common with all other members.

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