The Number Endings Problem

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Note: This was a problem I created for a middle schooler that seemed to keep spiraling. There are subproblems of varying levels of difficulty.

1 Special Number Endings

In base 10, squaring a whole number sometimes leaves pieces of that number behind. It's clear to us that squaring a number ending in zero always produces a number ending in zero, and squaring a number ending in five ends in five. Playing around a little, you could also convince yourself correctly that numbers ending in one square to numbers ending in one, and the same for six. However, these special things (we'll call them just "endings" throughout this problem set) can get a lot longer, and have some interesting properties.

Most interesting: *There is an infinitely long, nontrivial sequence of digits that, when squared, ends in itself. We can only ever know the back end of this number. And, in fact, there are two.*

1.1 Problem 1

Prerequisites: Persistence.

What combinations of two digits at the end of a number always see those two digits reappear at the end when squared? For example, $4100^2 = 16810000$, and similar for anything ending in 00. How about three digits? This requires only persistence or a little insight.

1.2 Problem 2

Prerequisites: Some algebra.

What do you notice about the two digit endings? If we have a two digit ending that works, what one digit endings must work? Why? What about finding a two digit ending if you know a three digit one? Does this continue indefinitely?

1.3 Problem 3

Prerequisites: Familiarity with programming.

Assume that, for every n there are *exactly four* endings of length n that square to themselves. Use this, and the previous solutions, to find the four 100-digit endings that square to themselves. **Hint**: It may be helpful to determine a method for predicting the suitable endings of length n starting with known endings of length $n-1$.

1.4 Problem 4

Prerequisites: Elementary Number Theory

Prove that there are exactly four endings that recreate themselves on squaring, no matter what the size is. So there is a trillion digit ending (well, four of them) of numbers that shows up again when you square a number that has it as an ending. There are even four $10^{trillion}$ digit endings!

2 Solutions

In general, the reader should first convince themselves of a rule of thumb we'll call the *Only Endings Matter Rule*:

Any number whose last digits are ending S will square to a number ending in S if and only if S^2 ends with S.

In other words, for our special endings, it really doesn't matter if any [more significant] digits come "before" our ending when evaluating how the endings behave upon squaring. This isn't really a special theorem - this is more a simple rule for those new to modular arithmetic. Note that this only works in bases like 10, 100, 1000, etc. since we're describing number endings using digit positions.

For example, it feels obvious that any number ending in 6 when squared ends in 6, but more formally, for any integer k ($k \in \mathbb{Z}$), $(10k+6)^2 = 100k^2 + 2 \times 6 \times 10k + 36 = 10(10k^2 + 2 \times 6 \times k + 3) + 6 \equiv 6 \mod 10$. That sentence means: whatever whole number $10 \times k$ composes the digits before a final 6 (whether that's 36, 106, 3423426, -4446...) doesn't matter for our purposes: the result of squaring will be "ten times some whole number", plus a 6 at the end.

Similar arguments work for any of the "endings" we discover. So if we're looking at the behavior of the two-digit ending "76", we don't need to check 176, 276, 376, etc. 76^2 does all the work for us.

2.1 Problem 1 solution

- By squaring out all of the number endings from 00 to 99, you can find endings 00, 01, 25, and 76.
- By squaring out all of the number endings from 000 to 999, you can find endings 000, 001, 625, and 376.

2.2 Problem 2 solution

In solving problem 1, you might cut down the space by noticing that any candidates *must* end in 0, 1, 5, or 6.

For if not, imagine the ending 47, when squared, left 47 as an ending, or $47^2 = 47 \mod 100$. In that case, adding any whole number of the form $10k$ and squaring would end with 7 too: $(10k + 47)^2 = 100k^2 + 2 * 47 * 10 + 47^2 = 10(10k^2 + 2 * 47) + 47^2$. The first term doesn't affect the last digit, and the last digit is 7 by hypothesis. Therefore, *any* ending $07, 17, ...47...97$ squares to end in 7. Therefore, 7 must be a single digit ending if this is so.

Of course, 7 does not end in itself when squared, so 47 can't be a possibility. This leaves only numbers ending in 0, 1, 5, 6 to investigate for 2-digit endings, and only numbers ending in 00, 01, 25, and 76 for three digit endings, and so on.

Swapping 47 with generic ending S, the above argument is easily generalized for any number of digits (size of the exponent for modulus 10^n). The upshot: *If digits $a_{n-1}a_{n-2}...a_0$ is an ending modulo 10^n, then $a_{n-2}...a_0$ is an ending modulo 10^{n-1}.* And, by contrapositive, *if $a_{n-2}...a_0$ isn't an ending, then $a_{n-1}a_{n-2}...a_0$ can't be either.* So let's call that the *Nesting Endings Rule.*

This will help us trim the space of solutions immensely.

2.3 Problem 3 solution

Here are the endings that square to themselves modulo 10^{100}:

- 00...0 (100 zeroes)
- 00....1 (99 zeroes, then 1)
- 3953007319108169802938509890062166509580863811
  000557423423230896109004106619977392256259918212890625
- 6046992680891830197061490109937833490419136188899
  9442576576769103890995893380022607743740081787109376

The first two should be obvious from the *Only Endings Matter Rule*, since $0^2 = 0, 1^2 = 1$.

The third ("the one ending in 5") needs to be found more systematically, and programming likely required in any case. However, brute force won't do it; using a computer to search through all numbers exhaustively may work for one, two, three, or even six digits, but 10^{100} is too large a search space. Two options occur to me:

2.3.1 Option 1: Using the Nesting Endings Rule

This is the most straightforward. We know that *if* a valid ending exists for ending size n, then it must contain an ending of size $n-1$. So armed with our ending S_{n-1} of size $n-1$, we find our next ending by the algorithm:

- Start with $S_1 = 5, n = 2$.
- Loop these:
  - Create new candidate $0S_{n-1}$. (Example: If the ending is 625, create 0625.)
  - Check if this candidate squares to itself modulo 10^n.
  - If so, move on to the next n ($n \leftarrow n+1$).
  - Otherwise, move on to the next candidate (in this case $(1S_n - 1)$), up through 9.
- *(If we're out of candidates, angry-mail fettermania@gmail.com. We shouldn't get here.)*

This works because:

- Checking up to 10 digits for 100 rounds is computationally achievable.
- The *Nesting Endings Rule* states that if there is an ending of greater size, it contains our last one.
- The problem statement assumed that we have four endings of every size. (We will prove this in another problem).
- Two endings that end in 0, 1, 5, or 6 of size n cannot "collide" and share an ending of size $n-1$. (Therefore, we have an unbroken string of endings at any size: the one that stops with 0, the one with 1, the one with 5, and the one with 6).

The no collisions fact can be proved directly. We cannot have x and $x + k10^n$ be distinct solutions modulo 10^{n+1}.

- We suppose $x^2 \equiv x \mod 10^{n+1}$ and $(x + k10^n)^2 \equiv (x + k10^n) \mod 10^{n+1}$
- This means $x^2 - x \equiv 0 \mod 10^{n+1}$ and $(x + k10^n)^2 - (x + k10^n) \equiv 0 \mod 10^{n+1}$

- Subtracting these, we get $k10^n(2x-1) \equiv 0 \mod 10^{n+1} \Rightarrow k(2x-1) \equiv 0 \mod 10$
- If x ends in 5, $2x-1$ cannot have factors 5 or 2. (The same is also true if x ends in 0, 1, or 6).
- Therefore k must be zero or have divisors 5 *and* 2, meaning it is 0 modulo 10.
- So, the only way x and $x+k10^n$ are both solutions is if they are the same modulo 10^{n+1}. So no collisions are possible.

2.3.2 Option 2: Computing the "5" answer directly

If constructing the answer by hand for the first few digits, you may notice a pattern:

- $5^2 = 25 \equiv 5 \mod 10$
- $25^2 = 625 \equiv 25 \mod 100$
- $625^2 = 390625 \equiv 625 \mod 1000$
- $625^2 = 390625 \equiv 0625 \mod 10000$
- $0625^2 = 390625 \equiv 90625 \mod 100000$
- $90625^2 = 8212890625 \equiv 890625 \mod 1000000$
- ...

So the next new digit seems to come from rightmost digit we 'dropped' last time when creating our next ending. Borrow that one, and it'll be the next prefix we're looking for in creating our subsequent ending. This works, though there's a more precise formulation. We can see that the formula $5^{2^{n-1}}$ will actually find our solution for modulus 10^n. In other words, just square our last solution:

- $n = 1$: $5^2 \equiv 5 \mod 10 \Rightarrow 5^2 - 5 \equiv 0 \mod 10 \Rightarrow 5(5-1) \equiv 0 \mod 10$
- $n = 2 : 25^2 \equiv 25 \mod 100 \Rightarrow 25^2 - 25 \equiv 0 \mod 100 \Rightarrow 5^2(5^2-1) = 5^2(5+1)(5-1) \equiv 0 \mod 100$
- $n = 3$: $625^2 \equiv 625 \mod 1000 \Rightarrow 625^2 - 625 \equiv 0 \mod 1000 \Rightarrow 5^4(5^4-1) = 5^4(5^2+1)(5^2-1) = 5^4(5^2+1)(5+1)(5-1) \equiv 0 \mod 1000$
- $n = 4$: $390625^2 \equiv 625 \mod 10000 \Rightarrow 390625^2 - 390625 \equiv 0 \mod 10000 \Rightarrow 5^8(5^8 - 1) = ... = 5^8(5^4+1)(5^2+1)(5+1)(5-1) \equiv 0 \mod 10000$. Note: The solution here, modulo 10000, is "0625".

We need to ensure that the candidate on the left hand size has the factors to ensure the modulus 10^n on the right hand side divides it. This works out to n 5s and n 2s. Our fives are easy, since $2^{n-1} \geq n$ for all $n \geq 1$. And we see that the cascading set of factors on the right gives us n even numbers, which takes care of our n 2's.

Naturally, computing $5^{2^{100}}$ is prohibitive. Since we're only looking at the last n digits, and in arithmetic modulo 10^n, those are the only digits that matter, we can simply take our last solution digits, square them, and take the result modulo 10^n for our next set of digits.

This number is so big it seems like it must be overkill. However, looking at the powers of 5 in a list, we do see a certain binary resemblance. As powers of 5^n proceed:

- The third digit (from right) cycles along the list of 2: $[6, 1]$, (looping back to the beginning when stepping over).
- The fourth digit from right cycles among 4: $[0, 3, 5, 8]$
- The fifth digit cycles among 8: $[9, 5, 6, 2, 4, 0, 1, 7]$
- The sixth digit cycles among 16: $[7, 8, 1, 7, 4, 5, 5, 8, 4, 2, 3, 6, 2, 0, 0, 3, 9]$

This suggests the "tumblers all line up" for endings to match only on powers of 2. *Note: There's probably another fun problem in here to create.*

2.4 What about the "6"?

We have our 00...0 and 00...1 cases, and the ability to compute our answer ending in 5. We're only left with the number ending in 6.

It turns out for digit length n, any solution S yields another partner solution $(1 - S) \mod 10^n$ (or, equivalently, $10^n + 1 - S$). To see this:

- $S^2 = S \mod 10^{n+1}$, by S being a solution.
- $(1 - S)^2 = 1 - 2S + S^2 = 1 - 2S + S = (1 - S) \mod 10^{n+1}$

The second equivalency on the second line follows from the first line.

However, we can't keep generating new solutions this way, since $1 - (1 - S) = S$. So every solution ending in "5" has a buddy solution ending in "6": $(5, 10 + 1 - 5 = 6), (25, 100 + 1 - 25 = 76), (625, 1000 + 1 - 625 = 376)$... This is how we found the 100-digit solution ending in "6" above.

2.5 Problem 4 Solution

We want to prove that there are exactly four unique solutions to $S^2 - S = S(S - 1) \equiv 0 \mod 10^n$. We know:

- If $S(S - 1) = 0$, then $S = 0$ or $S = 1$. This yields our two trivial solutions.

- Otherwise, $S(S-1)$ when factorized, must contain 2^n and 5^n if 10^n divides it.
- Since they differ by 1, the first and second factor cannot each contain a factor of 2.
- Since they differ by 1, the first and second factor cannot each contain a factor of 5.
- Therefore, S must be a multiple of 2^n and $S-1$ a multiple of 5^n, or the reverse.
- (1) In the first case, this means $S \equiv 0 \mod 2^n, S \equiv 1 \mod 5^n$.
- (2) In the second case, this means $S \equiv 0 \mod 5^n, S \equiv 1 \mod 2^n$.

The Chinese remainder theorem from Number Theory guarantees the existence and uniqueness of solutions for (1) and (2) up to $2^n * 5^n = 10^n$.

Therefore, we have exactly four solutions.